Determining Unitary Equivalence to a 3×3 Complex Symmetric Matrix from the Upper Triangular Form

Jay Daigle
Advised by Stephan Garcia

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Chapter 1

Introduction

A *complex symmetric matrix* (CSM) is a square complex matrix T such that $T = T^t$, where T^t denotes the transpose of the matrix T. However, since any given operator has many different matrix representaions, we wish to identify those linear operators that can be represented as a complex symmetric matrix with respect to some orthonormal basis. To do this we introduce the concept of unitary equivalence.

Definition. *Let U be a $n \times n$ matrix and let U^* denote the adjoint (i.e., conjugate transpose) of U. We say U is unitary if $UU^* = U^*U = I$.*

Definition. *Let T and S be $n \times n$ matrices. We say T and S are unitarily equivalent if there exists some unitary matrix U such that $U^*TU = S$.*

It turns out that two matrices are unitarily equivalent precisely when they represent the same operator with respect to two (possibly different) orthonormal bases. Thus the class of matrices we wish to study is precisely the class of matrices which are unitarily equivalent to a complex symmetric matrix (UECSM).

A powerful tool for analyzing such matrices (Theorem 1.1 below) comes from Garcia and Putinar [2, 3]. We first require a few preliminaries.

Definition. *A function $C : \mathbb{C}^n \to \mathbb{C}^n$ is a conjugation if*

1. *C is antilinear (i.e., $C(ax + y) = \overline{a}Cx + Cy$ for all $a \in \mathbb{C}, x, y \in \mathbb{C}^n$).*

2. *C is isometric (i.e., $\langle x, y\rangle = \langle Cy, Cx\rangle$ for all $x, y \in \mathbb{C}^n$).*

3. *C is an involution (i.e., $C^2 = I$).*

A simple example of a conjugation is the standard conjugation $J : \mathbb{C}^n \to \mathbb{C}^n$, which is defined by

$$J(x_1, \ldots, x_n) = (\overline{x_1}, \ldots, \overline{x_n}).$$

Definition. *Let T be a $n \times n$ complex matrix and let T^* denote its conjugate transpose. Let C be a conjugation on $\mathbb{C}^n$; then T is C-symmetric if $T = CT^*C$.*

We now have the following theorem:

Theorem 1.1. *Let T be a $n \times n$ complex matrix. T is UECSM if and only if T is C-symmetric for some conjugation C.*

Proof. Suppose T is C-symmetric and let $\{e_i\}$ be an orthonormal basis with $Ce_i = e_i$ (for a proof that such a basis exists, see [3, Lemma 1]). Then

$$\begin{aligned} \langle Te_i, e_j \rangle &= \langle e_i, T^* e_j \rangle \\ &= \langle Ce_i, T^* Ce_j \rangle \\ &= \langle CT^* Ce_j, e_i \rangle \text{ (by the isometric property)} \\ &= \langle Te_j, e_i \rangle. \end{aligned}$$

Thus, since $\langle Te_i, e_j \rangle$ is the jith entry of T when expressed with respect to the orthonormal basis $\{e_i\}$, we have represented T as a CSM with respect to this basis. Therefore the matrix T is UECSM.

Conversely, suppose T is UECSM. It must be complex symmetric with respect to some orthonormal basis $\{u_i\}$. Define C by extending $Cu_i = u_i$ antilinearly to all of $\mathbb{C}^n$. Then as above, we have $\langle Tu_i, u_j \rangle = \langle CT^* Cu_j, u_i \rangle$ for all i, j, but since T is symmetric with respect to $\{u_i\}$ we have $\langle Tu_i, u_j \rangle = \langle Tu_j, u_i \rangle$. Therefore $\langle CT^* Cu_i, u_j \rangle = \langle Tu_i, u_j \rangle$ for all i, j, and thus $T = CT^*C$. □

Though many examples of matrices which are UECSM are known, no complete classification exists. In this paper we complete a classification of the 3×3 matrices which are UECSM. Since unitary equivalence is an equivalence relation, we may pick one representative of each equivalence class and determine whether this matrix is UECSM. To select this representative we use Schur's Theorem (see [5] for details):

Theorem 1.2 (Schur)**.** *Given a $n \times n$ matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$ in any prescribed order, there is a unitary matrix $U \in M_n$ such that*

$$U^* A U = T = [t_{ij}]$$

is upper triangular, with diagonal entries $t_{ii} = \lambda_i$, $i = 1, \ldots, n$. That is, every square matrix A is unitarily equivalent to a triangular matrix whose diagonal entries are the eigenvalues of A in a prescribed order.

Thus, given a 3×3 matrix A, we know it is unitarily equivalent to a (not necessarily unique) matrix of the form

$$\begin{pmatrix} \lambda_1 & a & b \\ 0 & \lambda_2 & c \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{1.1}$$

where λ_i are the eigenvalues of A in any given order. In Chapter 2 we show that further simplifying assumptions can be made to yield a particularly easy-to-work-with form. We provide a complete classification of which matrices in this form are UECSM, allowing us to determine whether any given 3×3 matrix is UECSM.

Chapter 2

Technical Background

In this section we present several lemmas that allow us to conduct our classification. First we describe some simple categories of matrices that are known to be UECSM. The results in this section are known and can be found in [2, 3, 4].

We begin with a definition:

Definition. *We say a matrix is algebraic of degree n if its minimal polynomial is a degree n polynomial. In other words, T is of degree n if there is some polynomial f of degree n with $f(T) = 0$, but no polynomial g of degree less than n such that $g(T) = 0$.*

We now present a theorem due to Garcia and Wogen [4].

Theorem 2.1. *If T is a square matrix that is algebraic of degree two, then T is UECSM.*

This theorem has a few useful corollaries:

Corollary 2.2. *All 1×1 and 2×2 matrices are UECSM.*

Proof. The degree of the minimum polynomial of a $n \times n$ matrix is always $\leq n$. Thus every 1×1 and 2×2 matrix is algebraic of degree ≤ 2, and by Theorem 2.1 is UECSM. □

Corollary 2.3. *Every rank 1 matrix is UECSM.*

Proof. Every rank one operator has the form $Tx = \langle x, v\rangle u$ for some $u, v \in \mathbb{C}^n$. But then $T^2 = \langle(\langle x, v\rangle u), v\rangle u = \langle x, v\rangle\langle u, v\rangle u = \langle u, v\rangle T$, so we have $T^2 - \langle u, v\rangle T = 0$ and T is algebraic of degree two. Thus by Theorem 2.1, T is UECSM. □

In our attempt to classify the 3×3 matrices which are UECSM, it will be useful to be able to decompose our matrices into simpler matrices. The next lemma shows that we can, in some instances, do this:

Lemma 2.4. *Suppose $T = \bigoplus_{i=1}^{n} T_i$, where T_i is C_i symmetric for some C_i. Then T is C-symmetric for $C = \bigoplus_{i=1}^{n} C_i$.*

Proof. We prove the result for $n = 2$; the theorem then follows by induction. Let T_1 and T_2 be square complex matrices and C_1, C_2 conjugations such that $C_1 T_1 C_1 = T_1^*$ and $C_2 T_2 C_2 = T_2^*$. Then we claim $C = C_1 \oplus C_2$ is a conjugation. Antilinearity is trivially preserved by direct summation, and $C^2 = I$ since

$$\begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} = \begin{pmatrix} C_1 \circ C_1 & 0 \\ 0 & C_2 \circ C_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

To show C is isometric, let $x = (x_1, x_2)$ be in the domain of T. Since $(x_1, 0)$ and $(0, x_2)$ are orthogonal we have $\|(x_1,0)\|^2 + \|(0,x_2)\|^2 = \|(x_1,x_2)\|^2$. Thus

$$\begin{aligned} \|Cx\|^2 &= \left\| \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|^2 \\ &= \left\| \begin{pmatrix} C_1 x_1 \\ C_2 x_2 \end{pmatrix} \right\|^2 \\ &= \left\| \begin{pmatrix} C_1 x_1 \\ 0 \end{pmatrix} \right\|^2 + \left\| \begin{pmatrix} 0 \\ C_2 x_2 \end{pmatrix} \right\|^2 \\ &= \|x_1\|^2 + \|x_2\|^2 = \|x\|^2. \end{aligned}$$

Thus C is a conjugation. Now we show that T is C-symmetric:

$$\begin{aligned} CTC &= \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \\ &= \begin{pmatrix} C_1 T_1 C_1 & 0 \\ 0 & C_2 T_2 C_2 \end{pmatrix} = \begin{pmatrix} T_1^* & 0 \\ 0 & T_2^* \end{pmatrix} = T^*. \end{aligned}$$

Thus we have the desired result. □

Our final lemma allows us to make certain simplifying assumptions about the structure of our matrix:

Lemma 2.5. *If T is a C-symmetric matrix, then so is $aT + bI$ for all $a, b \in \mathbb{C}$.*

Proof. Consider the matrix $C(aT+bI)C$. We have

$$\begin{aligned} C(aT+bI)C &= C(aT)C + C(bI)C \\ &= \overline{a}CTC + \overline{b}C^2 \\ &= \overline{a}T^* + \overline{b}I \\ &= (aT+bI)^*, \end{aligned}$$

which is the desired result. □

Chapter 3

The Angle Test

Suppose that T is a C-symmetric matrix and that u is a generalized eigenvector with eigenvalue λ of order n (i.e., $(T-\lambda I)^n u = 0$ but $(T-\lambda I)^{n-1}u \neq 0$). We have

$$(T^* - \overline{\lambda} I)^k (Cu) = C((T-\lambda I)^k u)$$

for all k, and thus Cu is a generalized eigenvector of T^* with eigenvalue $\overline{\lambda}$ of order n. Furthermore, since C is an isometry we have $\|u\| = \|Cu\|$. Note that the case $k = 1$ implies that if u is an eigenvector of T, then Cu is an eigenvector of T^*.

Now suppose the eigenspace of T corresponding to the eigenvalue λ is one-dimensional, and u is a unit eigenvector with eigenvalue λ. Let v be a unit eigenvector of T^* with eigenvalue $\overline{\lambda}$. Then since each eigenspace is one-dimensional, these eigenvectors are unique up to multiplication by a unimodular constant, and we have

$$Cu = \alpha v$$

for some unimodular α.

A similar argument will hold for generalized eigenvectors as long as we have a canonical method for choosing a basis for the generalized eigenspace which is respected by C. Our next lemma provides such a canonical basis:

Lemma 3.1. *Let T be a $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_r$. Suppose that T has precisely one Jordan block for each eigenvalue. Let*

$$K = \ker(T - \lambda_i I)^k \ominus \ker(T - \lambda_i I)^{k-1}$$

and

$$\hat{K} = \ker(T^* - \overline{\lambda_i} I)^k \ominus \ker(T^* - \overline{\lambda_i} I)^{k-1}$$

such that $K \neq \{0\}$*, and let* $u \in K$ *and* $v \in \hat{K}$ *be unit vectors. Then* $Cu = \alpha v$ *for some unimodular constant* α*.*

Proof. It is clear from our discussion that Cu is a generalized eigenvector of T^* with eigenvalue $\overline{\lambda}$ of order k, and thus is an element of $\hat{K}$. Since there is only one Jordan block for $\overline{\lambda_i}$ in the Jordan form of T^* we have that $\dim \ker(T^* - \overline{\lambda_i} I)^k = k$, and thus $\dim \hat{K} = 1$. Then since v is non-zero it must span K^*, and so there is some scalar with $\alpha v = Cu$. But $\|v\| = \|Cu\|$, so $|\alpha| = 1$. □

We are now prepared to state and prove a necessary condition for a matrix T to be C-symmetric.

Lemma 3.2 (The Angle Test)**.** *Let* T *be a* $n \times n$ *matrix with* r *distinct eigenvalues* $\lambda_1, \ldots, \lambda_r$ *which is UECSM, and suppose* T *has precisely one Jordan block for each eigenvalue. Let*

$$K = \ker(T - \lambda_i I)^k \ominus \ker(T - \lambda_i I)^{k-1}$$

and

$$L = \ker(T - \lambda_j I)^l \ominus \ker(T - \lambda_j I)^{l-1},$$

and let $\hat{K}$ *and* $\hat{L}$ *be the analogous spaces for* T^**. Let* $u_1 \in K, u_2 \in L, v_1 \in \hat{K}, v_2 \in \hat{L}$*. Then*

$$|\langle u_1, u_2 \rangle| = |\langle v_1, v_2 \rangle|. \tag{3.1}$$

Proof. If K (or L) is $\{0\}$, then so is $\hat{K}$ (or $\hat{L}$). We can see that $u_1 = v_1 = 0$ (or $u_2 = v_2 = 0$) and so

$$|\langle u_1, u_2 \rangle| = |\langle v_1, v_2 \rangle| = 0.$$

So assume $K, L \neq \{0\}$. By the isometric property of C,

$$\langle u_1, u_2 \rangle = \langle Cu_2, Cu_1 \rangle.$$

By Lemma 3.1 there exist unimodular α_1, α_2 with $Cu_i = \alpha_i v_i$. Thus we have

$$\begin{aligned} \langle u_1, u_2 \rangle &= \langle Cu_2, Cu_1 \rangle \\ &= \langle \alpha_2 v_2, \alpha_1 v_1 \rangle \end{aligned}$$

$$= \overline{\alpha_1}\alpha_2\overline{\langle v_1, v_2\rangle}.$$

Since α_i are unimodular, taking the absolute value of both sides yields Equation (3.1). □

Lemma 3.2 provides a useful technique to prove that a given matrix with a single Jordan block for each eigenvalue is not UECSM: find a canonical basis for the generalized eigenspaces of T and T^* as described in Lemma 3.1, take the pairwise inner products of the basis eigenvectors, and check whether the norms are all the same. In particular, if a $n \times n$ matrix T has n distinct eigenvalues then the pairwise inner products of the eigenvectors must have matching norms. If equation (3.1) holds for all pairs of vectors in our canonical basis, we say that T *passes the Angle Test*.

While the condition in Lemma 3.2 is clearly a necessary condition for T to be UECSM, it is not clear that it is sufficient. However, Garcia showed [1] that the following stronger condition is sufficient when T has distinct eigenvalues:

Theorem 3.3. *Let T be a 3×3 matrix with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Let u_i be a unit eigenvector of T with eigenvalue λ_i, and let v_i be a unit eigenvector of T^* with eigenvalue $\overline{\lambda_i}$. Then T is UECSM if and only if T passes the Angle Test and there exist unimodular constants $b_{1,2}, b_{1,3}, b_{2,3}$ such that*

$$b_{i,j} = \frac{\langle u_i, u_j\rangle}{\langle v_j, v_i\rangle}$$

whenever $\langle v_j, v_i\rangle \neq 0$, and the matrix

$$B = \begin{pmatrix} 1 & b_{1,2} & b_{1,3} \\ \overline{b_{1,2}} & 1 & b_{2,3} \\ \overline{b_{1,3}} & \overline{b_{2,3}} & 1 \end{pmatrix}$$

has rank one.

In fact, a similar condition holds for any $n \times n$ matrix with n distinct eigenvalues. We omit the proof for brevity. However, we will prove the analogous (and somewhat more difficult) result for the case where a 3×3 matrix T has only two distinct eigenvalues.

Let T be a 3×3 C-symmetric matrix with two distinct eigenvalues. Without loss of generality we may, by Lemma 2.5, assume that the eigenvalues are 0 and 1, and that 0 has multiplicity two. Furthermore, we assume that $\dim(\ker T) = 1$, since whenever $\dim(\ker T) = 2$ then T is a rank one matrix and the case is trivial due to Corollary 2.3. Let

- u_0 be a unit eigenvector of T corresponding to the eigenvalue 0,
- u_{00} be a unit generalized eigenvector of T corresponding to the eigenvalue 0 that is orthogonal to u_0,
- u_1 be a unit eigenvector of T corresponding to the eigenvalue 1.

Furthermore, T^* also has eigenvalues 0 and 1 with the same multiplicity. Let

- v_0 be a unit eigenvector of T^* corresponding to the eigenvalue 0,
- v_{00} be a unit generalized eigenvector of T^* corresponding to the eigenvalue 0 that is orthogonal to v_0,
- v_1 be a unit eigenvector of T^* corresponding to the eigenvalue 1.

Lemma 3.1 tells us that C determines a triple of unimodular constants $\alpha_0, \alpha_{00}, \alpha_1$ with

$$\begin{aligned} Cu_0 &= \alpha_0 v_0, \\ Cu_{00} &= \alpha_{00} v_0, \\ Cu_1 &= \alpha_1 v_1. \end{aligned}$$

Thus for any i, j we have

$$\langle u_i, u_j \rangle = \langle Cu_j, Cu_i \rangle = \langle \alpha_j v_j, \alpha_i v_i \rangle = \overline{\alpha_i}\alpha_j \langle v_j, v_i \rangle.$$

Let B be the matrix given by

$$B = \begin{pmatrix} \overline{\alpha_0} \\ \overline{\alpha_{00}} \\ \overline{\alpha_1} \end{pmatrix} \begin{pmatrix} \alpha_0 & \alpha_{00} & \alpha_1 \end{pmatrix}.$$

Whenever $\langle v_j, v_i \rangle \neq 0$ we have

$$b_{i,j} = \overline{\alpha_i}\alpha_j = \frac{\langle u_i, u_j \rangle}{\langle v_j, v_i \rangle}.$$

Furthermore, we see that $b_{i,i} = 1$ for all i, and that $b_{i,j} = \overline{b_{j,i}}$ for all i, j. Thus the matrix B is self-adjoint. Also, note that $\text{rank}(B) = 1$ since

$$\text{colspace}(B) = \text{span}\left\{ \begin{pmatrix} \overline{\alpha_0} \\ \overline{\alpha_{00}} \\ \overline{\alpha_1} \end{pmatrix} \right\}.$$

Finally, note that since $Tu_{00} \in \ker(T)$, we have $Tu_{00} = au_0$ for some $a \in \mathbb{C}$. But

$$T^*v_{00} = CTCv_{00} = CT\alpha_{00}u_{00} = \overline{\alpha_{00}}Cau_0 = (\overline{a\alpha_{00}}\alpha_0)v_0.$$

Thus the coefficients of u_0 in Tu_{00} and v_0 in T^*v_{00} are related by a unimodular factor of $\overline{\alpha_{00}}\alpha_0\frac{\overline{a}}{a}$.

We are now prepared to state our main theorem.

Theorem 3.4. *Let T be a 3×3 matrix under the above hypotheses, with $Tu_{00} = au_0$. We have that T is C-symmetric if and only if there exists a set of unimodular constants $\alpha_0, \alpha_{00}, \alpha_1$ such that*

1. $\langle u_i, u_j \rangle = \overline{\alpha_i}\alpha_j \langle v_j, v_i \rangle$.
2. *The function C defined by letting $Cu_i = \alpha_i v_i$ and extending antilinearly to $\mathbb{C}^3$ is an involution.*
3. $T^*v_{00} = (\overline{a\alpha_{00}}\alpha_0)v_0$.

Proof. We wish to show that C is a conjugation and that $CT^*C = T$. We see that C is antilinear by construction and an involution by condition (2). We wish to show that it is an isometry. The u_i form a basis for our space, so let $x = \sum a_i u_i$. We have

$$\begin{aligned} \|x\|^2 &= \langle x, x \rangle \\ &= \left\langle \sum_{i \in \{0,00,1\}} a_i u_i, \sum_{j \in \{0,00,1\}} a_j u_j \right\rangle \\ &= \sum_{i,j \in \{0,00,1\}} a_i \left\langle u_i, \sum a_j u_j \right\rangle \\ &= \sum_{i,j \in \{0,00,1\}} a_i \sum \overline{a_j} \langle u_i, u_j \rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j\in\{0,00,1\}} a_i\overline{a_j}\langle u_i, u_j\rangle \\
&= \sum_{i,j\in\{0,00,1\}} a_i\overline{a_j\alpha_i}\alpha_j\langle v_j, v_i\rangle \\
&= \sum_{i,j\in\{0,00,1\}} a_i\overline{a_j}\langle \alpha_j v_j, \alpha_i v_i\rangle \\
&= \sum_{i,j\in\{0,00,1\}} a_i\overline{a_j}\langle Cu_j, Cu_i\rangle \\
&= \left\langle \sum_{j\in\{0,00,1\}} \overline{a_j}Cu_j, \sum_{i\in\{0,00,1\}} \overline{a_i}Cu_i \right\rangle \\
&= \langle Cx, Cx\rangle \\
&= \|Cx\|^2.
\end{aligned}
$$

Thus C is indeed a conjugation.

We now wish to show that $CT^*C = T$. Since both maps are linear we only need to show that they agree on the basis $\{u_0, u_{00}, u_1\}$. Note that since C is an involution we have $Cv_i = \alpha_i u_i$. For u_0 and u_1 we have

$$
\begin{aligned}
CT^*Cu_i &= CT^*\alpha_i v_i \\
&= \overline{\alpha_i}C\overline{\lambda_i}v_i \\
&= \overline{\alpha_i}\lambda_i\alpha_i u_i \\
&= \lambda_i u_i \\
&= Tu_i.
\end{aligned}
$$

Now consider CT^*Cu_{00}. We get

$$
\begin{aligned}
CT^*Cu_{00} &= \overline{\alpha_{00}}CT^*v_{00} \\
&= \overline{\alpha_{00}}C(\overline{a\alpha_{00}}\alpha_0)v_0 \\
&= \overline{\alpha_{00}}a\alpha_{00}\overline{\alpha_0}Cv_0 \\
&= a\overline{\alpha_0}\alpha_0 u_0 \\
&= au_0 \\
&= Tu_{00}.
\end{aligned}
$$

Thus $CT^*C = T$ and we have that T is C-symmetric. $\square$

Chapter 4

Breaking Down the Problem

4.1 Our Approach

By Schur's Theorem (Theorem 1.2), any 3×3 matrix T is unitarily equivalent to a matrix of the form shown in (1.1). Thus a matrix is UECSM if and only if its upper triangular form is UECSM. By Lemma 2.5 we may, without loss of generality, subtract a multiple of the identity to ensure one eigenvalue is 0. If there is more than one eigenvalue we may further divide by a scalar to obtain a matrix with an eigenvalue of 1. Based on the number of distinct eigenvalues we have the following three cases:

Case 1: One distinct eigenvalue.

We may assume T has the form

$$T = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.1}$$

Case 2: Two distinct eigenvalues.

We may assume T has the form

$$T = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.2}$$

Case 3: Three distinct eigenvalues.

We may assume T has the form

$$T = \begin{pmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & \lambda \end{pmatrix} \tag{4.3}$$

for some $\lambda \notin \{0, 1\}$.

4.2 The One Eigenvalue Case

Proposition 4.1. *A 3×3 matrix T in the upper triangular form given in (4.1) is UECSM if and only if one of the following holds:*

1. $a = 0$.

2. $c = 0$.

3. $|a| = |c|$, $a, c \neq 0$.

Proof. The proof breaks down into three cases.

Case 1: $a = 0$ or $c = 0$.

We have

$$T = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad T = \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In either case, T is a rank one matrix and thus UECSM by Corollary 2.3.

Case 2: $a, c \neq 0$, $|a| = |c|$.

Define an antilinear map $C : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$C \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{u} \\ 0 & \frac{\bar{a}}{cu} & 0 \\ u & 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{pmatrix}$$

where u is a unimodular constant such that $uc \in \mathbb{R}$. Then we have

$$\begin{aligned} CTC\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= CT\begin{pmatrix} u\bar{z} \\ \frac{\bar{a}y}{cu} \\ u\bar{x} \end{pmatrix} = C\begin{pmatrix} \frac{|a|\bar{y}}{cu} + bu\bar{x} \\ cu\bar{x} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \bar{a}x\frac{\bar{c}\bar{u}}{cu} \\ |u|\bar{b}x + |a|\frac{yu}{\bar{c}\bar{u}} \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{a}x \\ \bar{b}x + |c|\frac{y}{c} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ \bar{a} & 0 & 0 \\ \bar{b} & \bar{c} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}. \end{aligned}$$

Therefore $CTC = T^*$ and thus T is UECSM.

Case 3: $a, c \neq 0$, $|a| \neq |c|$.

Let e_1, e_2, e_3 be the standard basis vectors for $\mathbb{C}^3$. We can see that T has precisely one one-dimensional eigenspace for the eigenvalue 0, spanned by e_1. Similarly, T^* has precisely one one-dimensional eigenspace, spanned by e_3. By Lemma 3.1, if T is C-symmetric then $Ce_1 = \alpha e_3$ for some $|\alpha| = 1$. Since C is an involution, this gives us

$$e_1 = C\alpha e_3 = \overline{\alpha} C e_3$$

and thus $Ce_3 = \alpha e_1$. We see that e_2 is orthogonal to e_1 and e_3, so since C is an isometry we must have Ce_2 orthogonal to $Ce_1 = \alpha e_3$ and $Ce_3 = \alpha e_1$. Thus Ce_2 must be some scalar multiple of e_2, and in fact we have $Ce_2 = \beta e_2$ where $|\beta| = 1$ since C is norm-preserving.

Now $Te_2 = ae_1$. But

$$\begin{aligned} Te_2 &= CT^*Ce_2 \\ &= CT^*(\beta e_2) \\ &= \overline{\beta} C(\bar{c}e_3) \\ &= \alpha\overline{\beta}\bar{c}e_1 \end{aligned}$$

and thus $a = \alpha\overline{\beta}c$. Since $|\alpha| = |\beta| = 1$ this gives us $|a| = |c|$, a contradiction. Thus in this case T is not UECSM. $\square$

4.3 The Two Eigenvalue Case

Proposition 4.2. *A 3×3 matrix T in the upper triangular form given in (4.2) is UECSM if and only if one of the following holds:*

1. $a = 0$.

2. $b = c = 0$.

3. $a, c \neq 0$ *and* $|b + ac|^2 = |c|^2 + |c|^4$.

Proof. The proof breaks down into five cases.

Case 1: $a = 0$. We have

$$T = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus T is a rank one matrix, and is UECSM by Corollary 2.3.

Case 2: $b = c = 0$

It is clear that T is the direct sum of a 2×2 matrix and a 1×1 matrix:

$$T = \left(\begin{array}{cc|c} 0 & a & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right).$$

By Corollary 2.2, every 2×2 matrix and every 1×1 matrix is UECSM, and the direct sum of matrices which are UECSM is also UECSM by Lemma 2.4. Thus T is UECSM.

Case 3: $c = 0$ and $a, b \neq 0$.

We have

$$T = \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We see that T has a one-dimensional eigenspace with eigenvalue 1, and some eigenspace with eigenvalue 0. Since T has 1-dimensional nullspace, the eigenspace with eigenvalue 0 cannot be 2-dimensional. Thus T has one one-dimensional eigenspace with eigenvalue 0, and one one-dimensional eigenspace with eigenvalue 1. These spaces are spanned by the following unit vectors:

$$\begin{aligned} u_0 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} && \text{with eigenvalue} \quad 0, \\ u_1 &= \begin{pmatrix} \frac{b}{\sqrt{|b|^2+1}} \\ 0 \\ \frac{1}{\sqrt{|b|^2+1}} \end{pmatrix} && \text{with eigenvalue} \quad 1. \end{aligned}$$

Note that $\langle u_0, u_1 \rangle = \frac{\bar{b}}{\sqrt{|b|^2+1}}$.

Similarly, T^* has two one-dimensional eigenspaces:

$$\begin{aligned} v_0 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} && \text{with eigenvalue} \quad 0, \\ v_1 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} && \text{with eigenvalue} \quad 1. \end{aligned}$$

Note that $\langle v_0, v_1 \rangle = 0$.

But since $b \neq 0$ we have $|\langle u_0, u_1 \rangle| \neq |\langle v_0, v_1 \rangle|$, and T fails the Angle Test of Lemma 3.2. Thus T is not UECSM.

Case 4: $a, c \neq 0$.

We have

$$T = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

We use the conditions of Theorem 3.4: T is UECSM if and only if there exist unimodular $\alpha_0, \alpha_{00}, \alpha_1$ such that

1. $\langle u_i, u_j \rangle = \overline{\alpha_i} \alpha_j \langle v_j, v_i \rangle$.
2. The function C defined by letting $Cu_i = \alpha_i v_i$ and extending antilinearly is an involution.
3. $T^* v_{00} = (\overline{a\alpha_{00}}\alpha_0) v_0$, where a is as in T.

It is clear that if such α_i exist then the matrix

$$B = \begin{pmatrix} \overline{\alpha_0} \\ \overline{\alpha_{00}} \\ \overline{\alpha_1} \end{pmatrix} \begin{pmatrix} \alpha_0 & \alpha_{00} & \alpha_1 \end{pmatrix}$$

has rank one. Conversely, if there exists some matrix B such that $b_{i,i} = 1$, $|b_{i,j}| = 1$, B is self-adjoint, and whenever $\langle v_j, v_i \rangle \neq 0$ we have

$$b_{i,j} = \frac{\langle u_i, u_j \rangle}{\langle v_j, v_i \rangle},$$

then this matrix will factor as above and we will have a triple $(\alpha_0, \alpha_{00}, \alpha_1)$ that satisfies condition (1).

Looking now at the normalized eigenvectors of T and T^*, we get that

$$\begin{aligned} u_0 &= (1, 0, 0), \\ u_{00} &= (0, 1, 0), \\ u_1 &= \frac{1}{\sqrt{|b+ac|^2 + |c|^2 + 1}}(b + ac, c, 1), \end{aligned}$$

and

$$\begin{aligned} v_0 &= \frac{1}{\sqrt{1+|c|^2}}(0, 1, -\bar{c}), \\ v_{00} &= \frac{1}{\sqrt{(1+|c|^2)(|ac+b|^2 + |c|^2 + 1)}}(1 + |c|^2, -c\overline{(ac+b)}, -\overline{(ac+b)}), \\ v_1 &= (0, 0, 1). \end{aligned}$$

Thus $\langle u_0, u_{00} \rangle = \langle v_{00}, v_0 \rangle = 0$, but the other pairs of eigenvectors are not orthogonal. Assuming that T is UECSM, the Angle Test gives us

$$\begin{aligned} |\langle u_0, u_1 \rangle| &= |\langle v_0, v_1 \rangle|, \\ \left| \frac{b+ac}{\sqrt{|b+ac|^2 + |c|^2 + 1}} \right| &= \left| \frac{c}{\sqrt{1+|c|^2}} \right|, \\ |b+ac|^2(1+|c|^2) &= |c|^2(|b+ac|^2 + |c|^2 + 1), \\ |b+ac|^2 &= |c|^2 + |c|^4, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned}|\langle u_{00}, u_1\rangle| &= |\langle v_{00}, v_1\rangle|,\\ \frac{|c|}{\sqrt{|b+ac|^2+|c|^2+1}} &= \frac{|b+ac|}{\sqrt{(1+|c|^2)(|ac+b|^2+|c|^2+1)}},\\ |c|^2(1+|c|^2) &= |b+ac|^2,\\ |c|^2+|c|^4 &= |b+ac|^2,\end{aligned}$$

which is the same as (4.4). Thus the conditions of the Angle Test are satisfied if and only if (4.4) holds.

Now consider the matrix

$$B = \begin{pmatrix} 1 & x & \frac{\langle u_0,u_1\rangle}{\langle v_1,v_0\rangle} \\ \overline{x} & 1 & \frac{\langle u_{00},u_1\rangle}{\langle v_1,v_{00}\rangle} \\ \frac{\langle u_1,u_0\rangle}{\langle v_0,v_1\rangle} & \frac{\langle u_1,u_{00}\rangle}{\langle v_{00},v_1\rangle} & 1 \end{pmatrix},$$

where the x is a free unimodular constant that we get since $\langle v_{00}, v_0\rangle = 0$. It is clear that this matrix has rank one only if

$$x = \tfrac{\langle u_0,u_1\rangle}{\langle v_1,v_0\rangle}\tfrac{\langle u_1,u_{00}\rangle}{\langle v_{00},v_1\rangle}.$$

But as long as we also have

$$\left|\tfrac{\langle u_1,u_0\rangle}{\langle v_0,v_1\rangle}\right| = \left|\tfrac{\langle u_{00},u_1\rangle}{\langle v_1,v_{00}\rangle}\right| = 1$$

then we will have $\operatorname{rank}(B) = 1$. Thus as long as (4.4) holds there exists a triple that satisfies condition (1). In particular, if we set

$$\begin{pmatrix} \alpha_0 & \alpha_{00} & \alpha_1 \end{pmatrix} = \begin{pmatrix} 1 & \tfrac{\langle u_0,u_1\rangle}{\langle v_1,v_0\rangle}\cdot\tfrac{\langle u_1,u_{00}\rangle}{\langle v_{00},v_1\rangle} & \tfrac{\langle u_0,u_1\rangle}{\langle v_1,v_0\rangle} \end{pmatrix}$$

then the conditions in (1) are satisfied. Assuming that (4.4) holds, we get

$$\begin{aligned}\alpha_0 &= 1,\\ \alpha_{00} &= \frac{\overline{b+ac}}{1+|c|^2}\cdot\frac{-\sqrt{1+|c|^2}}{c}\cdot\frac{c}{1+|c|^2}\cdot\frac{\sqrt{1+|c|^2}^3}{-\overline{(b+ac)}} = 1,\\ \alpha_1 &= \frac{-\overline{(b+ac)}}{c\sqrt{1+|c|^2}^3},\end{aligned}$$

and further we see that

$$\begin{aligned} u_1 &= \frac{1}{1+|c|^2}(b+ac, c, 1), \\ v_{00} &= \frac{1}{\sqrt{1+|c|^2}^3}(1+|c|^2, -c\overline{(ac+b)}, -\overline{(ac+b)}). \end{aligned}$$

Now assume the preceding conditions are met and let C be defined as in (2), by extending the map $Cu_i = \alpha_i v_i$ antilinearly. To see if C is an involution we calculate Ce_i where e_i are the standard basis vectors. We see that $e_1 = u_0$ and $e_2 = u_{00}$, so $Ce_1 = \alpha_1 v_1$ and $Ce_2 = \alpha_{00} v_{00}$. To compute Ce_3 we use

$$\begin{aligned} e_3 &= (0,0,1) \\ &= \sqrt{|b+ac|^2+|c|^2+1}u_1 - (b+ac)u_0 - cu_{00} \\ &= (1+|c|^2)u_1 - (b+ac)u_0 - cu_{00} \text{ by (4.4)}, \end{aligned}$$

which implies that

$$\begin{aligned} Ce_3 &= (|c|^2+1)Cu_1 - \overline{(b+ac)}Cu_0 - \overline{c}Cu_{00} \\ &= (|c|^2+1)\alpha_1 v_1 - \overline{(b+ac)}\alpha_0 v_0 - \overline{c}\alpha_{00} v_{00} \\ &= \frac{-1}{\sqrt{(1+|c|^2)^3}}(\overline{c}(1+|c|^2), \overline{b+ac}, \tfrac{\overline{b+ac}}{c}) \end{aligned}$$

We know that C is an involution if $Cv_i = \alpha_i u_i$ for all i. We can see that $v_1 = e_3$ so we need

$$\frac{-1}{\sqrt{(1+|c|^2)^3}}(\overline{c}(1+|c|^2), \overline{b+ac}, \tfrac{\overline{b+ac}}{c}) = \alpha_1 u_1.$$

But we have

$$\begin{aligned} \alpha_1 u_1 &= \frac{\langle u_0, u_1\rangle}{\langle v_1, v_0\rangle} \cdot \frac{1}{1+|c|^2}(b+ac, c, 1) \\ &= \frac{\overline{b+ac}}{1+|c|^2} \cdot \frac{-\sqrt{1+|c|^2}}{c} \cdot \frac{1}{1+|c|^2}(b+ac, c, 1) \\ &= \frac{-1}{\sqrt{(1+|c|^2)^3}}\left(\tfrac{|b+ac|^2}{c}, \overline{b+ac}, \tfrac{\overline{b+ac}}{c}\right) \end{aligned}$$

$$\begin{aligned} &= \frac{-1}{\sqrt{(1+|c|^2)^3}}\left((1+|c|^2)\overline{c}, \overline{b+ac}, \tfrac{\overline{b+ac}}{c}\right) \\ &= Ce_3 \end{aligned}$$

For v_0 we have

$$v_0 = \frac{1}{\sqrt{1+|c|^2}}(e_2 - \overline{c}e_3).$$

Using our definition of C on the basis vectors we get

$$\begin{aligned} Cv_0 &= \frac{1}{\sqrt{1+|c|^2}}\left(\alpha_{00}v_{00} - c\frac{-1}{\sqrt{(1+|c|^2)^3}}(\overline{c}(1+|c|^2), \overline{b+ac}, \tfrac{\overline{b+ac}}{c})\right) \\ &= \frac{\left((1+|c|^2)(1+|c|^2), -c\overline{(ac+b)} + c\overline{(b+ac)}, \overline{(b+ac)} - \overline{(ac+b)}\right)}{(1+|c|^2)^2} \\ &= (1,0,0) \\ &= u_0. \end{aligned}$$

Finally, we consider v_{00}. We see that

$$v_{00} = \frac{1}{\sqrt{1+|c|^2}^3}(1+|c|^2)e_1 - c\overline{(ac+b)}e_2 - \overline{(ac+b)}e_3,$$

and so

$$Cv_{00} = \frac{1}{\sqrt{1+|c|^2}^3}(1+|c|^2)\alpha_0 v_0 - \overline{c}(ac+b)\alpha_{00}v_{00} - (ac+b)Ce_3.$$

Taking each of these terms in turn, we have

$$\begin{aligned} (1+|c|^2)\alpha_0 v_0 &= \frac{1}{\sqrt{1+|c|^2}^3}(0, (1+|c|^2)^2, -\overline{c}(1+|c|^2)^2), \\ \overline{c}(ac+b)\alpha_{00}v_{00} &= \frac{1}{\sqrt{1+|c|^2}^3}(\overline{c}(ac+b)(1+|c|^2), -|c|^2|ac+b|^2, -\overline{c}|ac+b|^2), \\ (ac+b)Ce_3 &= \frac{-1}{\sqrt{1+|c|^2}^3}(\overline{c}(ac+b)(1+|c|^2), |ac+b|^2, \tfrac{|ac+b|^2}{c}). \end{aligned}$$

So we see that

$$Cv_{00} = \frac{1}{(1+|c|^2)^3}(0, (1|c|^2)^3, -\overline{c}(1+2|c|^2+|c|^4) + \overline{c}(1+|c|^2)^2)$$

$$= (0,1,0) = \alpha_{00}u_{00}.$$

Thus for any u_i, we get

$$C^2u_i = C\alpha_i v_i = \overline{\alpha_i}\alpha_i u_i = u_i,$$

and we see that C^2 is the identity on the basis formed by $\{u_0, u_{00}, u_1\}$. Thus C is an involution, and condition (2) of Theorem 3.4 is satisfied.

Finally, we wish to show that condition (3) of Theorem 3.4 holds. We have

$$Tu_{00} = (a,0,0) = au_0,$$

so we wish to show that

$$T^*v_{00} = \overline{a}v_0.$$

But

$$\begin{aligned} T^*v_{00} &= \frac{1}{\sqrt{1+|c|^2}^3}(0, (1+|c|^2)\overline{a}, -(1+|c|^2)\overline{ac} \\ &= \overline{a}\frac{1}{\sqrt{1+|c|^2}}(0,1,-c) \\ &= av_0. \end{aligned}$$

Thus T meets the conditions of Theorem 3.4 if and only if $|b+ac|^2 = |c|^2+|c|^4$, and we have the desired result. □

4.4 The Three Eigenvalue Case

Proposition 4.3. *Let $T \in M_{3\times 3}(\mathbb{C})$ be in the upper triangular form given in* (4.3)*. Then T is UECSM if and only if one of the following holds:*

1. $a = b = 0$.

2. $b = c = 0$.

3. $a = c = 0$.

4. $a, c, b - ac \neq 0$,

$$\frac{(ac + b(\lambda-1))\overline{a}}{c\lambda + c|a|^2 + \overline{a}b(\lambda - 1)} = \frac{a(|c|^2 + |\lambda|^2 - \lambda) - b\overline{c}}{(1 + |\frac{\lambda}{b-ac}|^2(1+|a|^2))^2(a|c|^2 - b\overline{c})},$$

$$\begin{aligned}
\frac{\overline{a}(|a|^2\overline{c} + a\overline{b}(\overline{\lambda} - 1) + \overline{c\lambda})}{(1+|a|^2)\overline{(ac+b(\lambda-1))}} &= \frac{a(|c|^2+|\lambda|^2-\lambda) - b\overline{c}}{(1+|\frac{\lambda-1}{c}|^2)(a|c|^2 - b\overline{c})}, \\
\frac{\overline{a}|c|^2 - \overline{b}c}{\overline{a}(|c|^2+|\lambda|^2-\overline{\lambda}) - \overline{b}c} &= \frac{(ac+b(\lambda-1))(|a|^2\overline{c} + a\overline{b}(\overline{\lambda}-1) + \overline{c\lambda})}{a\lambda(\lambda-1)(1+|\frac{c}{\lambda-1}|^2 + |\frac{ac+b(\lambda-1)}{\lambda^2-\lambda}|^2)\overline{(\lambda^2-\lambda)}}.
\end{aligned}$$

Proof. The proof breaks down into seven cases.

Case 1: $a = b = 0$

It is clear that T is the direct sum of a 1×1 matrix and a 2×2 matrix, and thus UECSM:

$$T = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 1 & c \\ 0 & 0 & \lambda \end{array}\right).$$

Case 2: $b = c = 0$.

Similarly, T is the direct sum of a 2×2 matrix and a 1×1 matrix, and thus UECSM:

$$T = \left(\begin{array}{cc|c} 0 & a & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & \lambda \end{array}\right).$$

Case 3: $a = c = 0$, $b \neq 0$.

We have

$$T = \begin{pmatrix} 0 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

If wee define U by

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

we see that U is a unitary matrix. Furthermore,

$$U^*TU = \left(\begin{array}{cc|c} 0 & b & 0 \\ 0 & \lambda & 0 \\ \hline 0 & 0 & 1 \end{array}\right),$$

and thus T is unitarily equivalent to the direct sum of a 2×2 matrix and a 1×1 matrix, and UECSM.

Case 4: $c = 0$ and $a, b \neq 0$.

We have

$$T = \begin{pmatrix} 0 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

We can see that T has three one-dimensional eigenspaces, spanned by the following unit eigenvectors:

$$\begin{aligned}
u_0 &= (1,0,0) && \text{with eigenvalue } 0,\\
u_1 &= \left(\tfrac{|a|}{\sqrt{|a|^2+1}}, \tfrac{|a|}{a\sqrt{|a|^2+1}}, 0\right) && \text{with eigenvalue } 1,\\
u_\lambda &= \left(\tfrac{|b|}{\sqrt{\lambda^2+|b|^2}}, 0, \tfrac{\lambda|b|}{b\sqrt{\lambda^2+|b|^2}}\right) && \text{with eigenvalue } \lambda.
\end{aligned}$$

Similarly, T^* has three corresponding eigenspaces with corresponding unit eigenvectors:

$$\begin{aligned}
v_0 &= \left(\tfrac{\lambda}{\sqrt{(|a|^2+1)\lambda^2+|b|^2}}, \tfrac{-\lambda\bar{a}}{\sqrt{(|a|^2+1)\lambda^2+|b|^2}}, \tfrac{-\bar{b}}{\sqrt{(|a|^2+1)\lambda^2+|b|^2}}\right),\\
v_1 &= (0,1,0),\\
v_\lambda &= (0,0,1).
\end{aligned}$$

By Lemma 2.4 we know that if T is C-symmetric, then

$$|\langle u_0, u_\lambda\rangle| = |\langle v_0, v_\lambda\rangle|,$$

and computing these inner products gives us

$$\tfrac{|b|}{\sqrt{\lambda^2+|b|^2}} = \tfrac{|-\bar{b}|}{\sqrt{(|a|^2+1)\lambda^2+|b|^2}},$$

which implies that either $b = 0$, or

$$\lambda^2 + |b|^2 = (|a|^2+1)\lambda^2 + |b|^2$$

and thus $a = 0$, which is a contradiction. Thus T is not UECSM.

Case 5: $a = 0$, $b, c \neq 0$.

By an argument similar to the argument in Case 4, T is not UECSM. We have

$$T = \begin{pmatrix} 0 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & \lambda \end{pmatrix}.$$

Now T has three one-dimensional eigenspaces, spanned by the following unit eigenvectors:

$$\begin{aligned} u_0 &= (1,0,0) && \text{with eigenvalue } 0, \\ u_1 &= (0,1,0) && \text{with eigenvalue } 1, \\ u_\lambda &= \frac{1}{\sqrt{1 + \left|\frac{c}{\lambda-1}\right|^2 + \left|\frac{b}{\lambda}\right|^2}} \left(\tfrac{b}{\lambda}, \tfrac{c}{\lambda-1}, 1\right) && \text{with eigenvalue } \lambda. \end{aligned}$$

Similarly, T^* has three corresponding one-dimensional eigenspaces spanned by corresponding unit eigenvectors:

$$\begin{aligned} v_0 &= \frac{1}{\sqrt{1 + \left|\frac{\lambda}{b}\right|^2}} \left(-\tfrac{\overline{\lambda}}{\overline{b}}, 0, 1\right), \\ v_1 &= \frac{1}{\sqrt{1 + \left|\frac{\lambda-1}{c}\right|^2}} \left(0, \tfrac{1-\overline{\lambda}}{\overline{c}}, 1\right), \\ v_\lambda &= (0,0,1). \end{aligned}$$

By Lemma 2.4 we know that if T is C-symmetric, then

$$|\langle u_0, u_1 \rangle| = |\langle v_0, v_1 \rangle|,$$

and thus we have

$$0 = \frac{1}{\sqrt{\left(1 + \left|\frac{\lambda}{b}\right|^2\right)\left(1 + \left|\frac{\lambda-1}{c}\right|^2\right)}} > 0,$$

which is a contradiction. Thus T is not UECSM.

Case 6: $a, c \neq 0, b = ac$.

It is clear that T has three one-dimensional eigenspaces, spanned by the following unit eigenvectors:

$$u_0 = (1,0,0) \qquad \text{with eigenvalue } 0,$$
$$u_1 = \frac{1}{\sqrt{1+|a|^2}}(a,1,0) \qquad \text{with eigenvalue } 1,$$
$$u_\lambda = \frac{1}{\sqrt{|ac|^2+|c|^2+|\lambda-1|^2}}(ac,c,\lambda-1) \qquad \text{with eigenvalue } \lambda.$$

Similarly, T^* has three corresponding one-dimensional eigenspaces spanned by corresponding unit eigenvectors:

$$\begin{aligned} v_0 &= \frac{1}{\sqrt{1+|a|^2}}(1,-\overline{a},0), \\ v_1 &= \frac{1}{\sqrt{|1-\lambda|^2+|c|^2}}(0,1-\overline{\lambda},\overline{c}), \\ v_\lambda &= (0,0,1)\,. \end{aligned}$$

We can compute that

$$|\langle u_0, u_\lambda\rangle| = \left|\frac{ac}{\sqrt{|ac|^2+|c|^2+|\lambda-1|^2}}\right| \neq 0$$

but

$$\langle v_0, v_\lambda\rangle = 0,$$

and thus T fails the Angle Test and is not UECSM.

Case 7: $a, c, b - ac \neq 0$.

We use the rank one condition given in Theorem 3.3. Simple computation gives us that T has eigenspaces spanned by the normalized eigenvectors

$$\begin{aligned} u_0 &= (1,0,0), \\ u_1 &= \frac{1}{\sqrt{1+|a|^2}}(a,1,0)\,, \\ u_\lambda &= \frac{1}{\sqrt{1+|\frac{c}{\lambda-1}|^2+|\frac{b(1-\lambda)-ac}{\lambda^2-\lambda}|^2}}\left(\tfrac{ac+b(\lambda-1)}{\lambda^2-\lambda},\tfrac{c}{\lambda-1},1\right). \end{aligned}$$

Similarly, T^* has eigenspaces spanned by the normalized eigenvectors

$$\begin{aligned} v_0 &= \frac{1}{\sqrt{1+|\frac{\lambda}{b-ac}|^2+|\frac{a\lambda}{b-ac}|^2}}\left(\frac{\lambda}{\overline{ac}-\overline{b}}, \frac{\overline{a\lambda}}{\overline{b}-\overline{ac}}, 1\right), \\ v_1 &= \frac{1}{\sqrt{1+|\frac{\lambda-1}{c}|^2}}\left(0, \frac{1-\overline{\lambda}}{\overline{c}}, 1\right), \\ v_\lambda &= (0,0,1). \end{aligned}$$

Computing the pairwise inner products of these eigenvectors in turn gives us

$$\begin{aligned} \langle u_0, u_1\rangle &= \frac{\overline{a}}{\sqrt{1+|a|^2}}, \\ \langle u_0, u_\lambda\rangle &= \frac{\overline{ac}+\overline{b}(\overline{\lambda}-1)}{(\overline{\lambda}^2-\overline{\lambda})\sqrt{1+|\frac{c}{\lambda-1}|^2+|\frac{b(1-\lambda)-ac}{\lambda^2-\lambda}|^2}}, \\ \langle u_1, u_\lambda\rangle &= \frac{|a|^2\overline{c}+a\overline{b}(\overline{\lambda}-1)+\overline{c\lambda}}{\sqrt{(1+|a|^2)(1+|\frac{c}{\lambda-1}|^2+|\frac{b(1-\lambda)-ac}{\lambda^2-\lambda}|^2)}(\overline{\lambda}^2+\overline{\lambda})}, \qquad (4.5) \\ \langle v_0, v_1\rangle &= \frac{\overline{a}(\overline{\lambda}-|\lambda|^2-|c|^2)+\overline{b}c}{\sqrt{1+|\frac{\lambda-1}{c}|^2}\sqrt{1+|\frac{\lambda}{b-ac}|^2+|\frac{a\lambda}{b-ac}|^2}(\overline{b}c-\overline{a}|c|^2)}, \\ \langle v_0, v_\lambda\rangle &= \frac{1}{\sqrt{1+|\frac{\lambda}{b-ac}|^2+|\frac{a\lambda}{b-ac}|^2}}, \\ \langle v_1, v_\lambda\rangle &= \frac{1}{\sqrt{1+|\frac{\lambda-1}{c}|^2}}. \end{aligned}$$

Given our hypotheses that $\lambda \neq 0, 1$ and $b-ac \neq 0$, these are all well-defined. So now consider the matrix

$$B = \begin{pmatrix} 1 & \frac{\langle u_0,u_1\rangle}{\langle v_1,v_0\rangle} & \frac{\langle u_0,u_\lambda\rangle}{\langle v_\lambda,v_0\rangle} \\ \frac{\langle u_1,u_0\rangle}{\langle v_0,v_1\rangle} & 1 & \frac{\langle u_1,u_\lambda\rangle}{\langle v_\lambda,v_1\rangle} \\ \frac{\langle u_\lambda,u_0\rangle}{\langle v_0,v_\lambda\rangle} & \frac{\langle u_\lambda,u_1\rangle}{\langle v_1,v_\lambda\rangle} & 1 \end{pmatrix}.$$

The conjugate symmetry of the inner product gives us that

$$\frac{\langle u_i, u_j\rangle}{\langle v_j, v_i\rangle} = \overline{\frac{\langle u_j, u_i\rangle}{\langle v_i, v_j\rangle}},$$

and so this matrix has the form of the matrix B from Theorem 3.3.

The matrix B is rank one if and only if the columnspace of B is one-dimensional, if and only if each column is a scalar multiple of each other column. Let B_i be the ith column of B; then if B is rank one we must have

$$\begin{aligned} B_1 &= \overline{\frac{\langle u_0, u_1\rangle}{\langle v_1, v_0\rangle}} B_2, \\ B_2 &= \overline{\frac{\langle u_1, u_\lambda\rangle}{\langle v_\lambda, v_1\rangle}} B_3, \\ B_3 &= \frac{\langle u_0, u_\lambda\rangle}{\langle v_\lambda, v_0\rangle} B_1. \end{aligned} \tag{4.6}$$

The system in (4.6) implies nine equations, three of which are trivially true. The other six are given by

$$\left|\frac{\langle u_0, u_1\rangle}{\langle v_1, v_0\rangle}\right| = 1, \tag{4.7}$$

$$\left|\frac{\langle u_1, u_\lambda\rangle}{\langle v_\lambda, v_1\rangle}\right| = 1, \tag{4.8}$$

$$\left|\frac{\langle u_0, u_\lambda\rangle}{\langle v_\lambda, v_1\rangle}\right| = 1, \tag{4.9}$$

$$\frac{\langle u_0, u_1\rangle\langle u_\lambda u_0\rangle}{\langle v_1, v_0\rangle\langle v_0, v_\lambda\rangle} = \frac{\langle u_\lambda, u_1\rangle}{\langle v_1, v_\lambda\rangle}, \tag{4.10}$$

$$\frac{\langle u_0, u_1\rangle\langle u_1, u_\lambda\rangle}{\langle v_1, v_0\rangle\langle v_\lambda, v_1\rangle} = \frac{\langle u_0, u_\lambda\rangle}{\langle v_\lambda, v_0\rangle}, \tag{4.11}$$

$$\frac{\langle u_1, u_\lambda\rangle\langle u_\lambda, u_0\rangle}{\langle v_\lambda, v_1\rangle\langle v_0, v_\lambda\rangle} = \frac{\langle u_1, u_0\rangle}{\langle v_0, v_1\rangle}. \tag{4.12}$$

Note first that satisfying the rank-one condition implies satisfying the Angle Test because of equations (4.7), (4.8), and (4.9). Further, note that if we assume (4.10), (4.11), and (4.12), then substituting (4.11) into (4.10) gives

$$\frac{\langle u_\lambda, u_1\rangle}{\langle v_1, v_\lambda\rangle} = \frac{\langle u_0, u_1\rangle}{\langle v_1, v_0\rangle}\overline{\frac{\langle u_0, u_1\rangle\langle u_1, u_\lambda\rangle}{\langle v_1, v_0\rangle\langle v_\lambda, v_1\rangle}} = \frac{\langle u_\lambda, u_1\rangle}{\langle v_1, v_\lambda\rangle}\left|\frac{\langle u_0, u_1\rangle}{\langle v_1, v_0\rangle}\right|$$

and thus (4.7) holds. A similar argument gives that (4.8) and (4.9) hold as well, so the conditions of Theorem 3.3 are met if and only if we have (4.10), (4.11), and (4.12). Rearranging these equations gives

$$\begin{aligned}
\frac{\langle u_0, u_1\rangle\overline{\langle u_0, u_\lambda\rangle}}{\overline{\langle u_1, u_\lambda\rangle}} &= \frac{\overline{\langle v_0, v_1\rangle}\langle v_0, v_\lambda\rangle}{\langle v_1, v_\lambda\rangle} \\
\frac{\langle u_0, u_1\rangle\langle u_1, u_\lambda\rangle}{\langle u_0, u_\lambda\rangle} &= \frac{\overline{\langle v_0, v_1\rangle\langle v_1, v_\lambda\rangle}}{\overline{\langle v_0, v_\lambda\rangle}} \\
\frac{\overline{\langle u_0, u_\lambda\rangle}\langle u_1, u_\lambda\rangle}{\overline{\langle u_0, u_1\rangle}} &= \frac{\langle v_0, v_\lambda\rangle\overline{\langle v_1, v_\lambda\rangle}}{\langle v_0, v_1\rangle}
\end{aligned}$$

Plugging in values from Equation (4.5) gives the desired condition. □

This completes our classification of the 3×3 matrices from the upper triangular form. Given a 3×3 upper triangular matrix, these results allows us to easily determine whether it is UECSM. Since every matrix is equivalent to an upper triangular matrix, this is in effect a classification of all 3×3 matrices which are UECSM.

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