Math 114 Practice Test 1 Solutions

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Problem 1.

(a) Directly from the definition of a limit, compute with proof $\lim_{x\to -2} \frac{x}{x+4}$ **Solution:** We guess -1. Let $\epsilon > 0$ and let $\delta \leq \underline{1, \epsilon/2}$. Then if $0 < |x+2| < \delta$, we compute

$$\left| \frac{x}{x+4} + 1 \right| = \left| \frac{2x+4}{x+4} \right| = \frac{2|x+2|}{|x+4|}$$

And we compute

$$|x+4| = |(x+2)+2| \geq 2 - |x+2| > 2 - \delta \geq 1$$

by the reverse triangle inequality. So

$$\left| \frac{x}{x+4} + 1 \right| = \frac{2|x+2|}{|x+4|} < \frac{2\delta}{1} < \epsilon.$$

(b) Directly from the definition, compute with proof $\lim_{x\to 3} \frac{2x^2-10x+12}{x-3}$.
Solution: Let $\epsilon > 0$ and set $\delta \leq \underline{\epsilon/2}$. Then if $0 < |x-3| < \delta$ then

$$\begin{aligned} \left| \frac{2x^2-10x+12}{x-3} - 2 \right| &= \left| \frac{2(x-3)(x-2)}{x-3} - 2 \right| \\ &= |2(x-2) - 2| = 2|x-3| < 2\delta \leq \epsilon. \end{aligned}$$

Problem 2.
Let

$$f(x) = \begin{cases} 5 & x < -1 \\ 2 & x > -1 \end{cases}$$

(a) Directly from the definition, compute with proof $\lim_{x\to 1} f(x)$.
Solution: Let $\epsilon > 0$ and set $\delta = \underline{2}$. Then if $0 < |x-1| < \delta$, we see that $x > -1$ and thus we have

$$|f(x) - 2| = |2-2| = 0 < \epsilon.$$

(b) Directly from the definition of a limit, prove that $\lim_{x\to -1} f(x)$ does not exist.
Solution: Set $\epsilon = 1$ and suppose $\delta > 0$. Suppose $\lim_{x\to -1} f(x) = L$. Then set $x_1 = -1 + \delta/2, x_2 = -1 - \delta/2$, and we have

$$\begin{aligned} \epsilon &> |f(x_1) - L| = |f(-1+\delta/2) - L| = |2 - L| \\ \epsilon &> |f(x_2) - L| = |f(-1-\delta/2) - L| = |5 - L| \\ 2\epsilon &> |L-2| + |5-L| \geq |L - 2 + 5 - L| = |3| = 3 \end{aligned}$$

Thus we have $3 < 2\epsilon = 2$ which is impossible.

Problem 3.

Let

$$g(x) = \begin{cases} x-3 & x<3 \\ 2x+1 & x>3 \end{cases}$$

(a) Directly from the definition, compute with proof $\lim_{x\to 0} g(x)$.

Solution: Let $\epsilon > 0$ and set $\delta \le \underline{3, \epsilon}$. Then if $0 < |x-0| < \delta$ we have $x < 3$ and thus we compute

$$|g(x)+3| = |x-3+3| = |x| < \delta \le \epsilon.$$

Thus the limit is -3.

(b) Directly from the definition of a limit, prove that $\lim_{x\to 3} g(x)$ does not exist.

Solution:

Suppose $\lim_{x\to 3} g(x) = L$. Set $\epsilon = \underline{3}$ and let $\delta > 0$. Let $x_1 = 3 - \delta/2$ and $x_2 = 3 + \delta/2$. Then we have

$$\begin{aligned} \epsilon &> |g(x_1) - L| = |3 - \delta/2 - 3 - L| = |-\delta/2 - L| = |L + \delta/2| \\ \epsilon &> |g(x_2) - L| = |2(3+\delta/2) + 1 - L| = |7 + \delta - L| = |L - 7 - \delta| \\ 2\epsilon &> |-\delta/2 - L| + |L - 7 - \delta| \ge |-7 - 3\delta/2| = 7 + 3\delta/2 > 7. \end{aligned}$$

Since $\epsilon = 3$ this gives us $6 > 7$, which impossible. So no such limit exists.

Problem 4. (a) Directly from the definition, prove that $\lim_{x\to -4} \frac{x}{4+x} = \pm\infty$.

Solution: Let $N > 0$ and set $\delta \le \underline{1, 3/N}$. Then if $0 < |x+4| < \delta$, we have

$$\left|\frac{x}{4+x}\right| = \frac{|x|}{|x+4|} > \frac{|x|}{\delta}.$$

We observe that $|x| = |x+4-4| \ge 4 - |x+4| \ge 4 - \delta \ge 3$, which gives us

$$\left|\frac{x}{4+x}\right| = \frac{|x|}{|x+4|} > \frac{|x|}{\delta} \ge \frac{3}{\delta} > \frac{3}{3/N} = N.$$

(b) Directly from the definition, prove that $\lim_{x\to -2} \frac{x}{(x+2)^2} = -\infty$.

Solution: Let $N > 0$, and set $\delta \le \underline{1, 1/\sqrt{N}}$. Then if $0 < |x+2| < \delta$, then

$$\begin{aligned} &(x+2)^2 < \delta^2 \le 1/N \\ &\frac{1}{(x+2)^2} > \frac{1}{\delta^2} \ge N \\ &x = (x+2) - 2 < \delta - 2 \le 1 - 2 = -1 \\ &\frac{x}{(x+2)^2} < -N \qquad \text{(sign flips because } -1 < 0\text{).} \end{aligned}$$

Problem 5. Compute the following limits, showing each step and naming each limit law you use.

(a)

$$\lim_{x\to 4} \sqrt{x^2 - x - 3} + \frac{2}{x}$$

Solution:

$$\begin{aligned}
\lim_{x\to 4}\sqrt{x^2-x-3}+\frac{2}{x} &= \lim_{x\to 4}\sqrt{x^2-x-3}+\lim_{x\to 4}\frac{2}{x} && \text{Additivity}\\
&= \sqrt{\lim_{x\to 4}x^2-x-3}+\lim_{x\to 4}\frac{2}{x} && \text{Exponents}\\
&= \sqrt{\lim_{x\to 4}x^2-\lim_{x\to 4}x-\lim_{x\to 4}3}+\lim_{x\to 4}\frac{2}{x} && \text{Additivity}\\
&= \sqrt{\left(\lim_{x\to 4}x\right)^2-\lim_{x\to 4}x-\lim_{x\to 4}3}+\lim_{x\to 4}\frac{2}{x} && \text{Exponents}\\
&= \sqrt{\left(\lim_{x\to 4}x\right)^2-\lim_{x\to 4}x-\lim_{x\to 4}3}+\frac{\lim_{x\to 4}2}{\lim_{x\to 4}x} && \text{Quotients}\\
&= \sqrt{(4)^2-4-\lim_{x\to 4}3}+\frac{\lim_{x\to 4}2}{4} && \text{Identity}\\
&= \sqrt{(4)^2-4-3}+\frac{2}{4} && \text{Constants}\\
&= \sqrt{16-4-3}+\frac{2}{4}=3+\frac{1}{2} && \text{Arithmetic}
\end{aligned}$$

(b)

$$\lim_{x\to 1}\frac{x^2+4x-5}{x-1}$$

Solution:

$$\begin{aligned}
\lim_{x\to 1}\frac{x^2+4x-5}{x-1} &= \lim_{x\to 1}\frac{(x+5)(x-1)}{x-1} && \text{Arithmetic}\\
&= \lim_{x\to 1}x+5 && \text{Almost Identical Functions}\\
&= \lim_{x\to 1}x+\lim_{x\to 1}5 && \text{Additivity}\\
&= 1+\lim_{x\to 1}5 && \text{Identity}\\
&= 1+5 && \text{Constants}\\
&= 6
\end{aligned}$$