Problem 1.

Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x\to -2} \frac{x^2+6x+8}{2(x+4)(x+2)} =$$

Solution:

$$\lim_{x\to -2} \frac{x^2+6x+8}{2(x+4)(x+2)} = \lim_{x\to -2} \frac{(x+4)(x+2)}{2(x+4)(x+2)} = \lim_{x\to -2} \frac{1}{2} = 1/2.$$

(b)

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x}$$

Solution:

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x} = \lim_{x\to 9} \frac{(3-\sqrt{x})(3+\sqrt{x})}{(9-x)(3+\sqrt{x})} = \lim_{x\to 9} \frac{9-x}{(9-x)(3+\sqrt{x})} = \lim_{x\to 9} \frac{1}{3+\sqrt{x}} = 1/6.$$

(c)

$$\lim_{x\to -\infty} \frac{3x^3+\sqrt[3]{x}}{\sqrt{9x^6+2x^2+1}+x}$$

Solution:

$$\begin{aligned}
\lim_{x\to -\infty} \frac{3x^3+\sqrt[3]{x}}{\sqrt{9x^6+2x^2+1}+x} &= \lim_{x\to -\infty} \frac{3x^3/x^3+\sqrt[3]{x}/x^3}{\sqrt{9x^6+2x^2+1}/(-\sqrt{x^6})+x/x^3} \\
&= \lim_{x\to -\infty} \frac{3+x^{-8/3}}{-\sqrt{9+2x^{-4}+x^{-6}}+x^{-2}} \\
&= \lim_{x\to -\infty} \frac{3}{-\sqrt{9}} = -1.
\end{aligned}$$

(d)

$$\lim_{x\to 1^+} \frac{|x-1|}{x-1} =$$

Solution: We note that when $x > 1$, $|x-1| = x-1$, so we have

$$\lim_{x\to 1^+} \frac{|x-1|}{x-1} = \lim_{x\to 1^+} \frac{x-1}{x-1} = \lim_{x\to 1^+} 1 = 1$$

Problem 2.

Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} =$$

Solution:

$$\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} = \lim_{x\to 1} \left(\frac{\sin(x-1)}{x-1}\right)^2 = \left(\lim_{x\to 1} \frac{\sin(x-1)}{x-1}\right)^2 = 1^2 = 1$$

by the small angle approximation.

(b)

$$\lim_{x\to -2} \frac{x^2+6x+9}{2(x+4)(x+2)} =$$

Solution: We know that $\lim_{x\to-2} x^2+6x+9 = 1$ and $\lim_{x\to-2} 2(x+4)(x+2) = 0$. So

$$\lim_{x\to -2} \frac{x^2+6x+9}{2(x+4)(x+2)} = \pm\infty.$$

Since $2(x+4)(x+2)$ can be either positive or negative near -2—it is negative for values just less than -2 and positive for values just greater—we can't do any better than this.

(c)

$$\lim_{x\to\pi} \frac{\sin(x)}{x} =$$

Solution:

$$\lim_{x\to\pi} \frac{\sin(x)}{x} = \frac{0}{\pi} = 0.$$

(d)

$$\lim_{x\to 3} \frac{x-5}{(x-3)^2} =$$

Solution:

$$\lim_{x\to 3} \frac{x-5}{(x-3)^2} = -\infty$$

since the top approaches -2 and the bottom approaches zero and is always positive.

Problem 3. (a) Using the Squeeze Theorem, show that

$$\lim_{x\to 3} \frac{x-3}{1+\sin^2\left(\frac{2\pi+e+7}{x-3}\right)} = 0.$$

Solution: Observe that since $-1 \le \sin(a) \le 1$ for any a, we have that $0 \le \sin^2(a) \le 1$ for any a, and thus $1 \le 1+\sin^2(a) \le 2$. Taking the reciprocal gives us $1/2 \le \frac{1}{1+\sin^2(a)} \le 1$ for any a, and in particular for $a = \frac{2\pi+e+7}{x-3}$. Taking absolute values and multiplying by $|x-3|$ gives

$$\left|\frac{x-3}{2}\right| \le \left|\frac{x-3}{1+\sin^2\left(\frac{2\pi+e+7}{x-3}\right)}\right| \le |x-3|.$$

By continuity, we can compute that $\lim_{x\to3} |(x-3)/2 = \lim_{x\to 3} |x-3| = 0$. So by the squeeze theorem we know that

$$\lim_{x\to 3} \left|\frac{x-3}{1+\sin^2\left(\frac{2\pi+e+7}{x-3}\right)}\right| = 0,$$

and thus

$$\lim_{x\to 3} \frac{x-3}{1+\sin^2\left(\frac{2\pi+e+7}{x-3}\right)} = 0.$$

(b) Let

$$g(x) = \begin{cases} \frac{x^2-1}{x-1} & x > 0 \\ x^2+1 & x < 0 \end{cases}$$

If possible, define an extension of g that is continuous at all real numbers. **Solution:** g fails to be defined at 2 points: 0 and 1. We see that

$$\lim_{x\to 1} g(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 2$$

so we wish to set $g_F(1) = 2$. (Alternatively, we can just replace the $\frac{x^2-1}{x-1}$ with an $x+1$).

At 0, we need to compute the two one-sided limits. We have

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} x^2+1 = 1$$
$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{x^2-1}{x-1} = \frac{-1}{-1} = 1.$$

Thus the discontinuity is removeable, and we want to set $g_F(0) = 1$. Thus our continuous extension is

$$g_F(x) = \begin{cases} x+1 & x > 0 \\ 1 & x = 0 \\ x^2+1 & x < 0 \end{cases} = \begin{cases} x+1 & x \geq 0 \\ x^2+1 & x \leq 0 \end{cases}$$

Problem 4. (a) Show that the polynomial $x^4 - 6x - 2$ has two real roots, that is, there are two (different!) real numbers a and b such that $a^4 - 6a - 2 = b^4 - 6b - 2 = 0$.

Solution: Set $f(x) = x^4 - 6x - 2$; since this is a polynomial function it must be continuous. We compute:

$$f(0) = -2 \qquad\qquad f(-1) = 5$$
$$f(1) = -7 \qquad\qquad f(2) = 2$$

We have $-2 < 0 < 5$, so by the Intermediate Value Theorem there is some a between -1 and 0 with $f(a) = 0$. Similarly, we have $-7 < 0 < 2$ so by the Intermediate Value theorem there is some b between 1 and 2 with $f(b) = 0$. Clearly a and b are different since $a < 0$ and $b > 1$, so a and b are two distinct roots to the polynomial $x^4 - 6x - 2$.

(b) Directly from the definition of derivative, compute $f'(x)$ if $f(x) = \sqrt{x+3}$.

Solution:

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h}$$
$$= \lim_{h\to 0} \frac{x+h+3-(x+3)}{h(\sqrt{x+h+3}+\sqrt{x+3})}$$
$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+3}+\sqrt{x+3})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h+3}+\sqrt{x+3}} = \frac{1}{2\sqrt{x+3}}.$$

Problem 5. Compute the following derivatives using only the definition of derivative.

(a) Derivative of $f(x) = x^2 + \sqrt{x}$ at $x = 2$.

Solution:

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\
&= \lim_{h\to 0} \frac{(2+h)^2+\sqrt{2+h}-2^2-\sqrt{2}}{h} \\
&= \left(\lim_{h\to 0} \frac{4h+h^2}{h}\right) + \left(\lim_{h\to 0} \frac{(\sqrt{2+h}-\sqrt{2})(\sqrt{2+h}+\sqrt{2})}{h(\sqrt{2+h}+\sqrt{2})}\right) \\
&= \left(\lim_{h\to 0} 4+h\right) + \left(\lim_{h\to 0} \frac{1}{\sqrt{2+h}+\sqrt{2}}\right) \\
&= 4+\frac{1}{2\sqrt{2}}.
\end{aligned}$$

(b) Derivative of $g(x) = \frac{1}{x+1}$ at $x = 1$.

Solution:

$$\begin{aligned}
g'(1) &= \lim_{h\to 0} \frac{g(1+h)-g(1)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{2+h}-\frac{1}{2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2}{4+2h}-\frac{2+h}{4+2h}}{h} \\
&= \lim_{h\to 0} \frac{-h}{h(4+2h)} \\
&= \lim_{h\to 0} \frac{-1}{4+2h} \\
&= \frac{-1}{4}.
\end{aligned}$$