5 Integration

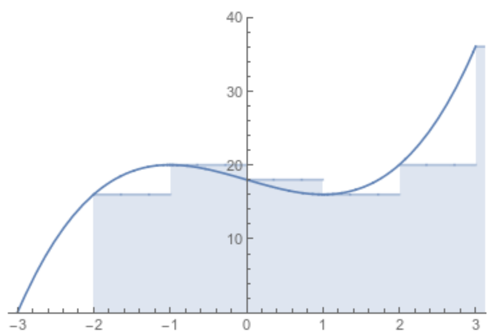
5.1 Riemann sums in multiple variables

Fundamentally, integrals are trying to add up all the value a function has in a given region. We do this by dividing the region up into a bucnh of subregions, estimating the total value in each subregion, and then adding these all back up.

In single-variable calculus we did this with a *Riemann Sum*. You might recall that we defined

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^n f(x_i^*)\Delta x = \lim_{n\to\infty} \sum_{i=1}^n f\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n}.$$

The basic idea here is that we divide the interval $[a,b]$ up into n subintervals. Then we pick some point x_i^* in the subinterval to represent the "average" value in that interval, and estimate the total value to be $f(x_i^*)\Delta x$. We graphically represent this by drawing a rectangle for every subinterval with height $f(x_i^*)$, and adding up the areas of the rectangles.

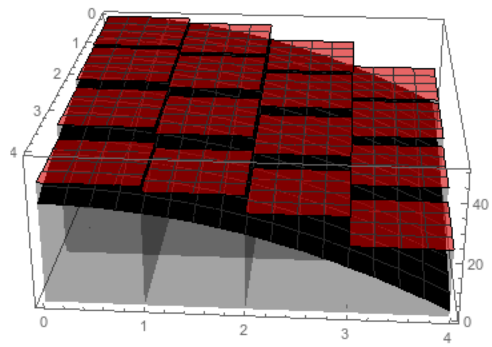


We'd like to do the same thing for a function of two or more variables. We'll stick with a two-variable function for now, and build the same picture. But since our function has two input variables, the geometry becomes three-dimensional. Rather than starting with an interval and dividing it into subintervals, we'll start with a rectangle and divide it into subrectangles.

Definition 5.1. Suppose $f(x,y)$ is continuous on a rectangle $R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$. Let (u_{ij}, v_{ij}) be any point in the ijth subrectangle. We define the *definite integral* of f over R to be

$$\int_R f\,dA = \lim_{\Delta x, \Delta y \to 0} \sum_{i,j} f(u_{ij}, v_{ij})\Delta x \Delta y.$$

If R is a non-rectangular region, we define $\int_R f\,dA$ similarly, except we ignore any subrectangle not contained in R. We can think of this as treating $f(u_{ij}, v_{ij}) = 0$ if (u_{ij}, v_{ij}) is not in R.



With the single-variable integral, we might worry that it matters *which* choice of value we take, but it turns out that that doesn't matter: in the limit they will converge to the same thing. The same is true in more variables.

Theorem 5.2. *If $f(x, y)$ is continuous and R is bounded, then $\int_R f dA$ converges, and the limit does not depend on the choices of (u_{ij}, v_{ij}).*

Sketch of Proof. If f is continuous on a closed and bounded region, then as Δx and Δy tend to zero, the difference between the maximum and minimum possible values of $f(x, y)$ within each rectangle tend to zero. Thus the largest possible sum and the smallest possible sum will converge to the same point; by the squeeze theorem, any intermediate sum will also converge. □

We can interpret this sum in a couple of different ways. One is volume. In the single-variable case, the integral estimates the (signed) area under the curve. In the multiple variable case, it estimates the volume under the surface given by the graph of the function.

Example 5.3. Suppose we want to estimate the area under the function $f(x, y) = 16 - 3x^2 - y^2$ on the rectangle with corners at $(0, 0)$ and $(2, 2)$. We can divide this up into four subrectangles, each of which is 1×1.

First let's get a definite overestimate, by always taking the highest point in each subrectangle. It's not too hard to see that for f, this will always be the point closest to the origin. So we have

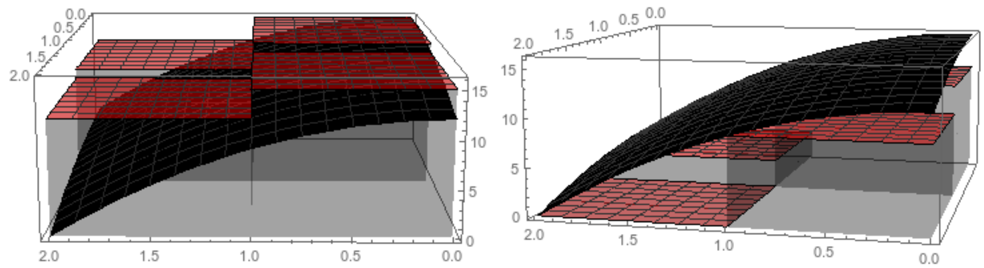
$$\int_R f da \approx f(0, 0) \cdot 1 + f(1, 0) \cdot 1 + f(0, 1) \cdot 1 + f(1, 1) \cdot 1 = 16 + 13 + 15 + 12 = 56.$$

We can also get an underestimate by taking the lowest value, which in this case will always be the upper-right point.

$$\int_R f da \approx f(1, 1) \cdot 1 + f(2, 1) \cdot 1 + f(1, 2) \cdot 1 + f(2, 2) \cdot 1 = 12 + 3 + 9 + 0 = 24.$$

So we can be pretty sure the volume is somewhere between 24 and 56. We would probably estimate something like $(24 + 56)/2 = 40$.

(If we compute the integral exactly, as we will learn in the next section, we will see that the integral is $\frac{128}{3} \approx 42.67$, so this estimate isn't too bad!)



"Volume under the surface" is a good way to interpret a 2-dimensional integral, but doesn't make much sense of a three-dimensional integral. (We can talk about the "hyper-volume" of the four-dimensional region, but that doesn't give much intuition since we can't really visualize hypervolumes).

Another way of understanding the integral is to think about averages. The integral $\int_R f dA$ is somehow computing the "total" value of f in the region. So we can also compute the *average* value of f in the region to be

$$\text{average} = \frac{1}{\text{Area}(R)} \int_R f dA.$$

This interpretation makes perfect sense in any number of variables we choose.

Definition 5.4. Suppose $f(x, y, z)$ is continuous on a region R, and let $(u_{ijk}, v_{ijk}, w_{ijk})$ be a point in the ijkth sub-prism. Then we define

$$\int_R f dV = \lim_{\Delta x, \Delta y, \Delta z \to 0} \sum_{i,j,k} f(u_{ijk}, v_{ijk}, w_{ijk}) \Delta x \Delta y \Delta z.$$

5.2 Iterated integrals

Computing multivariable integrals by writing out an expression for the Riemann sum and computing the limit is terrible. Fortunately we don't have to do that.

In single-variable calculus, we avoided doing the Riemann sum through the Fundamental Theorem of Calculus, which allowed us to evaluate an antiderivative on the endpoints of an interval, rather than summing the function on the whole interval. That is in fact possible to do here, but is somewhat complex, since the boundary of a two-dimensional region has infinitely many points. We'll return to this idea towards the end of class. But for right now, we'll do something much simpler.

When we wrote down the definition of a two-variable Riemann sum, we just said to add up the values for all the subrectangles; we didn't say anything about what order to add them up in. And as long as the sum is finite, this can't possibly matter.

For infinite sums, the order you add things up in can matter (see e.g. the Riemann Series Theorem if you want to know more about this). But fortunately, it turns out that in this case it does not.

Theorem 5.5 (Fubini)**.** *Let $R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\} = [a,b] \times [c,d]$, and let $f(x,y)$ be continuous on R. Then*

$$\int_R f dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

This means that rather than somehow doing the "whole" double integral, we can do two single-variable integrals in succession. And we already know how to do those!

Example 5.6. Let $R = \{(x,y) : 1 \leq x \leq 4, 0 \leq y \leq 3\}$ and let $f(x,y) = xy^2$. Then we can compute

$$\begin{aligned}\int_R f\,dA &= \int_1^4 \int_0^3 xy^2\,dy\,dx = \int_1^4 \left(xy^3/3|_0^3\right)\,dx \\ &= \int_1^4 9x\,dx = 9x^2/2|_1^4 = 72 - 9/2 = 135/2.\end{aligned}$$

Alternatively, we could compute:

$$\begin{aligned}\int_R f\,dA &= \int_0^3 \int_1^4 xy^2\,dx\,dy = \int_0^3 \left(x^2/2y^2|_1^4\right)\,dy \\ &= \int_0^3 8y^2 - y^2/2\,dy = \int_0^3 15y^2/2\,dy = 15y^3/6|_0^3 = 135/2.\end{aligned}$$

Notice we get the same answer with either order of integration.

Example 5.7. Suppose we have a building with a corrugated sine-wave roof. It is 6 meters wide and 8 meters long. The corners are 2 and 3 meters high, and along the length the sine wave oscillates four times. What is the volume of the building?

The height is given by $f(x,y) = 2 + x/6 + \sin(\pi y)$. Then the volume is given by

$$\begin{aligned}\int_0^6 \int_0^8 2 + x/6 + \sin(\pi y)\,dy\,dx &= \int_0^6 \left(2y + xy/6 - \cos(\pi y)/\pi\big|_0^8\right)\,dx \\ &= \int_0^6 16 + 4x/3 - 1/\pi - (0 + 0 - 1/\pi)\,dx \\ &= \int_0^6 16 + 4x/3\,dx = 16x + 2x^2/3|_0^6 = 96 + 24 = 120.\end{aligned}$$

Integrals of three-variable functions work exactly the same way that integrals of two variables work. We just have three iterated integrals instead of two.

Example 5.8. Suppose we have a box that has a 3 inch square base, and is 4 inches tall, and has a density of $1 + xy + yz + xz^2$ ounces per cubic inch. What is the total mass?

We want to compute the integral of $f(x,y,z) = 1 + xy + yz + xz^2$ over this rectangular box. So we compute

$$\begin{aligned}M &= \int_0^3 \int_0^3 \int_0^4 1 + xy + yz + xz^2\,dz\,dy\,dx = \int_0^3 \int_0^3 z + xyz + yz^2/2 + xz^3/3|_0^4\,dy\,dx \\ &= \int_0^3 \int_0^3 4 + 4xy + 8y + 64x/3\,dy\,dx = \int_0^3 4y + 2xy^2 + 4y^2 + 64xy/3|_0^3\,dx \\ &= \int_0^3 12 + 18x + 36 + 64x\,dx = 48x + 41x^2|_0^3 = 513.\end{aligned}$$

Thus the box has a mass of 513 ounces.

We can also use iterated integrals to integrate over non-rectangular (or non-box) regions. In this case we'll let x (say) vary from its minimum possible value to its maximum possible value; but for each x, the possible y values will depend on the current x value.

Example 5.9. Integrate the function $f(x,y) = xy$ over the triangle with corners at $(0,0)$, $(1,0)$, and $(1,3)$.

We have x varying from 0 to 1. The upper bound of the triangle is given by the line $y = 3x$, so the y bounds are from 0 to $3x$. Thus we have the double integral

$$\begin{aligned}\int_0^1 \int_0^{3x} xy\,dy\,dx &= \int_0^1 xy^2/2|_0^{3x}\,dx = \int_0^1 9x^3/2\,dx \\ &= 9x^4/8|_0^1 = 9/8.\end{aligned}$$

We could just as easily have done it the other way. y varies from 0 to 3, and x varies from $y/3$ to 1. So we have the double integral

$$\begin{aligned}\int_0^3 \int_{y/3}^1 xy\,dx\,dy &= \int_0^3 x^2y/2|_{y/3}^1\,dy = \int_0^3 (y/2 - y^3/18)\,dy \\ &= y^2/4 - y^4/72|_0^3 = 9/4 - 9/8 = 9/8.\end{aligned}$$

Thus we get the same answer integrating either way.

Example 5.10. Let's integrate the function $f(x, y) = y\sqrt{x}$ over the parallelogram with corners at $(0, 1), (0, 2), (1, 0), (1, 1)$.

We see that x varies from 0 to 1, and y varies from $1 - x$ to $2 - x$. So we have

$$\begin{aligned}
\int_0^1 \int_{1-x}^{2-x} y\sqrt{x}\, dy\, dx &= \int_0^1 2y^2\sqrt{x}/2|_{1-x}^{2-x}\, dx = \int_0^1 (2-x)^2\sqrt{x}/2 - (1-x)^2\sqrt{x}/2\, dx \\
&= \int_0^1 (4 - 4x + x^2 - 1 + 2x - x^2)\sqrt{x}/2\, dx = \int_0^1 3/2\sqrt{x} - x^{3/2}\, dx \\
&= x^{3/2} - 2/5x^{5/2}|_0^1 = 3/5.
\end{aligned}$$

Could we integrate the other way? Sure. But it's actually a big pain, since writing x as a function of y would have to go piecewise: we'd get something like

$$x = \begin{cases} 1 - y \leq x \leq 1 & y \leq 1 \\ 0 \leq x \leq 2 - y & 1 \leq y \leq 2 \end{cases}$$

So we'd have to set up and evaluate two separate integrals here, and get something like

$$\int_0^1 \int_{1-y}^1 y\sqrt{x}\, dx\, dy + \int_1^2 \int_0^{2-y} y\sqrt{x}\, dy\, dx.$$

Integrating by y and then x is very much the correct choice here.

Remark 5.11. Whenever setting up an iterated integral, remember that the final answer should be a number. Therefore the bounds of the outer integral should always be constants. The bounds on the inner integrals can depend on variables from integrals to the outside, but not on variables from integrals to the inside.

At each step, you should have one fewer variable to worry about (although possibly a more complex algebraic expression).

Example 5.12. Find the volume of the region bounded by $z = x+y, z = 10$, and the planes $x = 0, y = 0$.

We can set this up as a two-variable integral or as a three-variable integral. As a two-variable integral we'd need the region in the plane and the height. The solid exists over a region bounded by $0 \leq x \leq 10$ and $0 \leq y \leq 10-x$. Then the height is given by the difference between $z = 10$ and $z = x + y$, so we have $f(x, y) = 10 - x - y$. Then we get the integral

$$\begin{aligned}
\int_0^{10} \int_0^{10-x} 10 - x - y \, dy \, dx &= \int_0^{10} 10y - xy - y^2/2|_0^{10-x} \, x \\
&= \int_0^{10} 100 - 10x - 10x + x^2 - (100 - 20x + x^2)/2 \, dx \\
&= \int_0^{10} 50 - 10x + x^2/2 \, dx = 50x - 5x^2 + x^3/6|_0^{10} \\
&= 500 - 500 + 500/6 = 500/6.
\end{aligned}$$

But it's actually a bit more natural to express this as a triple integral. The volume of a region is just the integral of the function 1 over that region. So we can write

$$\begin{aligned}
V &= \int_0^{10} \int_0^{10-x} \int_{x+y}^{10} dz \, dy \, dx \\
&= \int_0^{10} \int_0^{10-x} z|_{x+y}^{10} \, dy \, dz \\
&= \int_0^{10} \int_0^{10-x} 10 - x - y \, dy \, dx.
\end{aligned}$$

This of course gets us the same answer as before, but is often a bit easier to think about.

Example 5.13. Let's find the average value of the function $f(x, y) = y(x - 1)$ on the region bounded by $y = x$ and $y = x^2$.

We can set up $0 \leq x \leq 1$ and $x^2 \leq y \leq x$. So we get the integral

$$\begin{aligned}
\int_0^1 \int_{x^2}^x y(x-1) \, dy \, dx &= \int_0^1 y^2/2(x-1)|_{x^2}^x \, dx = \int_0^1 x^3/2 - x^2/2 - (x^5/2 - x^4/2) \, dx \\
&= \int_0^1 -x^5/2 + x^4/2 + x^3/2 - x^2/2 \, dx = -x^6/12 + x^5/10 + x^4/8 - x^3/6|_0^1 \\
&= -1/12 + 1/10 + 1/8 - 1/6 = -1/40.
\end{aligned}$$

That gives us the total value of the function. We could also do the integral in the opposite order: we have $0 \leq y \leq 1$ and $y \leq x \leq \sqrt{y}$, and we get

$$\begin{aligned}
\int_0^1 \int_y^{\sqrt{y}} y(x-1)\,dx\,dy &= \int_0^1 x^2y/2 - xy|_y^{\sqrt{y}}\,dy = \int_0^1 y^2/2 - y^{3/2} - (y^3/2 - y^2)\,dy \\
&= \int_0^1 -y^3/2 + 3y^2/2 - y^{3/2}\,dy = -y^4/8 + y^3/2 - 2/5y^{5/2}|_0^1 \\
&= -1/8 + 1/2 - 2/5 = -1/40.
\end{aligned}$$

But we wanted the average, so we still need the area of the region. This is basically a single-variable calculus question: we integrate the height of the region over the interval. But we can also set it up as a multivariable integral: we just integrate the function "1" over the region. We get

$$\begin{aligned}
A &= \int_0^1 \int_{x^2}^x 1\,dy\,dx = \int_0^1 y|_{x^2}^x\,dx = \int_0^1 x - x^2\,dx \\
&= x^2/2 - x^3/3|_0^1 = 1/2 - 1/3 = 1/6.
\end{aligned}$$

(It's not an accident that the integral at the end of line 1 is exactly what you'd get by setting this up as a single-variable integral in Calculus 2).

Thus the total value is $-1/40$, and the area is $1/6$. So the average value of f on the region is

$$\text{average} = \frac{1}{\text{area}} \int_R f\,dA = 6(-1/40) = -3/20.$$

Example 5.14. Set up an integral to find the mass of a solid cone bounded by the xy plane and the cone $z = 4 - \sqrt{x^2+y^2}$, if the density is given by $\delta(x,y,z) = xz$.

We have the iterated integral

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^3}} \int_0^{4-\sqrt{x^2+y^2}} xz\,dz\,dy\,dx.$$

Example 5.15. Set up an integral to find the volume o the solid below the graph of $f(x,y) = 25 - x^2 - y^2$ and above the plane $z = 9$.

The two surfaces intersect where $x^2 + y^2 = 16$. We can either write the double integral

$$\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 16 - x^2 - y^2\,dy\,dx$$

or we can write the triple integral

$$\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_9^{25-x^2-y^2} dz\,dy\,dx$$

Example 5.16. Set up an integral to find the volume of the region in the first octant bounded by the coordinate planes, the plane $z = 3$, and the surface $z = x^2 + y^2$.

We can see we have z varying from 0 to 3. For each z, we have x varying from 0 to $\sqrt{z}$, and then y varying from 0 to $\sqrt{z - x^2}$. So we get the integral

$$\int_0^3 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-x^2}} 1 \, dy \, dx \, dz.$$

In most of these cases we have a few different options for how to set up the integral. So far these choices haven't mattered that much, but sometimes they matter a great deal.

Example 5.17.

$$\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1} \, dy \, dx.$$

The integral with respect to y is a huge pain, so we don't do it. We sketch the region: x goes from 0 to 6, and y goes from $x/3$ to 2. We can turn this around to say: y goes from 0 to 2, and x goes from 0 to $3y$. So we get

$$\begin{aligned}
\int_0^2 \int_0^{3y} x\sqrt{y^3+1} \, dx \, dy &= \int_0^2 \left(x^2/2\sqrt{y^3+1}\Big|_0^{3y} \right) dy \\
&= \int_0^2 \left(9y^2/2\sqrt{y^3+1} \right) dy \\
&= (y^3+1)^{3/2}|_0^2 = 27 - 1 = 26.
\end{aligned}$$

Example 5.18.

$$\begin{aligned}
\int_0^2 \int_y^2 e^{x^2} \, dx \, dy &= \int_0^2 \int_0^x e^{x^2} \, dy \, dx \\
&= \int_0^2 xe^{x^2} \, dx = e^{x^2}/2|_0^2 = e^4/2 - 1/2.
\end{aligned}$$

And sometimes, no matter what you do, the integral will be gross.

Example 5.19. Find the mass of the solid bounded by the xy plane, the yz plane, the xz plane, and the plane $x + 3y + 2z = 6$, if the density is given by $\delta(x, y, z) = x + z$.

We see that x varies from 0 to 6, and then z varies from 0 to $(6-x)/2$, and then y varies from 0 to $(6 - x - 2z)/3$. So we get

$$M = \int_0^6 \int_0^{3-x/2} \int_0^{2-x/3-2z/3} x + z \, dy \, dz \, dx = 27/2.$$

Example 5.20. Integrate $f(x,y) = x^2y$ over the upper half of the unit circle.

We have that $-1 \leq x \leq 1$ and $0 \leq y \leq \sqrt{1-x^2}$. So we get

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} x^2y\,dy\,dx &= \int_{-1}^{1} x^2y^2/2|_0^{\sqrt{1-x^2}}\,dx\\ &= \int_{-1}^{1} x^2(1-x^2)/2\,dx = \int_{-1}^{1} x^2/2 - x^4/2\,dx\\ &= x^3/6 - x^5/10|_{-1}^{1} = 1/6 - 1/10 + 1/6 - 1/10 = 2/15.\end{aligned}$$

Example 5.21. Integrate $f(x,y) = x^2y^2$ over the upper half of the unit circle. We have that $-1 \leq x \leq 1$ and $0 \leq y \leq \sqrt{1-x^2}$. So we get

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} x^2y^2\,dy\,dx &= \int_{-1}^{1} x^2y^3/3|_0^{\sqrt{1-x^2}}\,dx\\ &= \int_{-1}^{1} x^2(1-x^2)^{3/2}/3\,dx\end{aligned}$$

and this has suddenly become a huge mess—much worse than the previous problem. We can use trigonometric substitution plus some grindy arguments to find that this is equal to

$$\frac{1}{144}\left(x\sqrt{1-x^2}(-8x^4+14x^2-3)+3\arcsin(x)\right)\Big|_{-1}^{1} = \frac{\pi}{48},$$

but ultimately there's nothing we can do to this integral that will make it nice.

The fundamental problem in this last exampleis that since we're integrating over a circle, we have these $\sqrt{1-x^2}$ terms that we just can't get rid of.

Unless we develop a completely different approach to setting up integrals, that somehow is more compatible with circles.

5.3 Integrals in Polar Coordinates

Describing circles in Cartesian coordinates is fundamentally a bit awkward. It's much easier to describe a circle or circle-like region in terms of polar coordinates.

Definition 5.22. The *polar coordinates* of a point $P \in \mathbb{R}^2$ are a pair of numbers (r,θ), where r is the distance between P and the origin O, and θ is the angle between the vector $\vec{i}$ and the vector $\overrightarrow{OP}$.

We always choose these numbers so that r is positive, and $\theta \in [0, 2\pi)$.

Proposition 5.23. *Suppose (x,y) are the cartesian coordinates of a point P, and $(r\,theta)$ are the polar coordinates. Then:*

- $x = r\cos\theta$
- $y = r\sin\theta$
- $r = \sqrt{x^2+y^2}$
- $\theta = \pm\arctan y/x.$

Example 5.24. The polar equation for a circle of radius c is $r = c$. The closed disk of radius c is given by the set $\{(r,c) : 0 \leq r \leq c, 0 \leq \theta < 2\pi\}$. The Cartesian coordinates are $\{(x,y) : x^2+y^2 \leq c^2\}$.

The wedge of the closed unit disk in the first (upper-right) quadrant is $\{(r,\theta) : 0 \leq r \leq 1, 0 \leq \theta \leq \pi/2\}$. The Cartesian coordinates are $\{(x,y) : x \geq 0, y \geq 0, x^2+y^2 \leq 1.\}$

The set $\{(r,\theta) : 1 \leq r \leq 2, \pi \leq \theta \leq 3\pi/2\}$ is a wedge of an annulus with inner radius 1 and outer radius 2, in the third (lower-left) quadrant. The Cartesian coordinates here are $\{(x,y) : x \leq 0, y \leq 0, 1 \leq x^2+y^2 \leq 4\}$.

The polar equation for the line $y = 2x$ is $r\sin\theta = 2r\cos\theta$, which reduces to $\sin\theta = 2\cos\theta$.

Notice that all the circle equations become much simpler than their cartesian equivalents, but the line (and anything else rigid and rectangular) becomes much more complex.

We want to exploit this complexity reduction to make integrals of functions over circular regions easier. When we integrated over a rectangular region, we did this by dividing the region into rectangles. Using polar coordinates to integrate over a circular or wedge-like region, we'll divide the region into *subwedges*.

What is the area of a wedge? Each wedge is *roughly* a rectangle. (This is very rough, but in the limit it all washes out). The thickness of the rectangle is the change in the radius, so we call that dr. The width of the rectangle is *proportional* to the change in angle, but not equal to it: by definition, an arc of θ radians has a length of θr. Thus the width of our rectangle is $r\,d\theta$, the change in the angle times the actual radius.

This if we want to integrate a function in polar coordinates, we use the formula

$$I = \int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

Note the extra r in the formula! This is very important, and converts a number of integrals from "obnoxious" to "easy".

Example 5.25. Let's integrate $f(x,y) = x^2y$ over the upper half of the unit circle.

We see that this is a region given by $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi$. So we compute

$$\begin{aligned}
I &= \int_0^1 \int_0^\pi r^2 \cos^2 \theta r \sin \theta r \, d\theta \, dr \\
&= \int_0^1 \int_0^\pi r^4 \cos^2 \theta \sin \theta \, d\theta \, dr \\
&= \int_0^1 r^4 \frac{-1}{3} \cos^3 \theta |_0^\pi \, dr = \int_0^1 r^4 \frac{2}{3} \, dr \\
&= \frac{2}{15} r^5 |_0^1 = \frac{2}{15}.
\end{aligned}$$

Example 5.26. What about $f(x,y) = x^2y^2$ over that same region? We have

$$\begin{aligned}
I &= \int_0^1 \int_0^\pi r^2 \cos^2 \theta r^2 \sin^2 \theta r \, d\theta \, dr \\
&= \int_0^1 \int_0^\pi r^5 \cos^2 \theta \sin^2 \theta \, d\theta \, dr \\
&= \int_0^1 r^5 \left(\frac{\theta}{8} - \frac{1}{32} \sin(4\theta) \right) \Big|_0^\pi \, dr \\
&= \int_0^1 \frac{\pi r^5}{8} \, dr = \left. \frac{\pi r^6}{48} \right|_0^1 = \frac{\pi}{48}.
\end{aligned}$$

And while I didn't actually show the work to do that first antiderivative, it's a standard calc 2 trick—unlike the non-polar version, which is basically undoable.

Some functions also become much easier to integrate in polar coordinates.

Example 5.27. Integrate the function $f(x,y) = (x^2+y^2)^{-1/2}$ over the annulus with inner radius 1 and outer radius 2.

We have bounds $1 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. More importantly, we see that $f(x,y) = (x^2+y^2)^{-1/2} = (r^2)^{-1/2} = \frac{1}{r}$. Thus we have

$$\begin{aligned}
I &= \int_0^{2\pi} \int_1^2 \frac{1}{r} \, dr \, d\theta = \int_0^{2\pi} \ln |r| |_1^2 \, d\theta \\
&= \int_0^{2\pi} \ln 2 \, d\theta = 2\pi \ln 2.
\end{aligned}$$

Example 5.28. Let's find the area of the spiral that has thickness 1, and has inner radius going from 0 to 1 over one complete rotation.

$$\begin{aligned}
\int_0^{2\pi} \int_{\theta/(2\pi)}^{1+\theta/(2\pi)} r\,dr\,d\theta &= \int_0^{2\pi} \left.\frac{r^2}{2}\right|_{\theta/(2\pi)}^{1+\theta/(2\pi)} d\theta \\
&= \frac{1}{2}\int_0^{2\pi} (1+\theta/(2\pi))^2 - (\theta/(2\pi))^2\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 1+\theta/\pi + \theta^2/(4\pi^2) - \theta^2/(4\pi^2)\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 1+\theta/pi\,d\theta = \frac{1}{2}\left(\theta + \theta^2/(2\pi)|_0^{2\pi}\right) \\
&= \frac{1}{2}\left(2\pi + (2\pi)^2/(2\pi)\right) = 2\pi.
\end{aligned}$$

5.4 Cylindrical and Spherical Coordinates

We can extend this idea to three dimensions. There are two different ways to do this, which are suited to different types of regions.

Definition 5.29. The *cylindrical coordinates* of a point $P \in \mathbb{R}^3$ are a triple of numbers (r, θ, z), where r is the distance between the origin O and the projection of P into the xy plane; and θ is the angle between the vector $\vec{i}$ and the projection of $\overrightarrow{OP}$ into the xy plane; and z is the height.

We always choose these numbers so that r is positive, and $\theta \in [0, 2\pi)$.

Proposition 5.30. *Suppose (x, y, z) are the cartesian coordinates of a point P, and $(r\,theta, h)$ are the polar coordinates. Then:*

- $x = r\cos\theta$
- $y = r\sin\theta$
- $r = \sqrt{x^2+y^2}$
- $\theta = \pm\arctan y/x$
- $z = h.$

We can work out the integral formula here, just like we did for polar integrals. We divide our region into three-dimensional wedges—imagine a wedge of cheese. Each wedge is *roughly* a rectangular prism, as in polar integrals. The area of the base of the wedge is still $r\,dr\,d\theta$, and the height is dz, so when we do our integrals in cylindrical coordinates, we integrate $f(r, \theta, z) r\,dr\,d\theta\,dz$.

Example 5.31. Integrate xz over wedge cut from cylinder 4 cm high and 6 cm in radius, angle $\pi/6$ above x axis.

$$\int_0^4 \int_0^6 \int_0^{\pi/6} r\cos\theta z r \, d\theta \, dr \, dz = 288$$

Example 5.32. Integrate the function xyz over the cone bounded by $0 \leq z \leq 4$ and $x^2 + y^2 = z^2$ and the plane $z = 0$.

$$\int_0^4 \int_0^z \int_0^{2\pi} r^3 \cos\theta \sin\theta z \, d\theta \, dr \, dz = 0$$

Example 5.33. Set up an integral in spherical coordinates to find the volume inside the unit sphere.

$$\int_0^{2\pi} \int_{-1}^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} dz \, dr \, d\theta.$$

As we can see, cylindrical coordinates are still pretty unsuited to describing actual spheres. For those, we want to use a different coordinate system entirely.

Definition 5.34. The *spherical coordinates* of a point $P \in \mathbb{R}^3$ are a triple of numbers (ρ, θ, ϕ), where ρ is the distance between the origin O and the point P; θ is the angle between the vector $\vec{i}$ and the projection of $\overrightarrow{OP}$ into the xy plane; and ϕ is the angle between the vector $\overrightarrow{OP}$ and the vector $\vec{k}$.

We always choose these numbers so that ρ is positive, $\theta \in [0, 2\pi)$, and $\phi \in [0, \pi]$.

Proposition 5.35. *Suppose (x, y, z) are the cartesian coordinates of a point P, and $(r\, theta, h)$ are the polar coordinates. Then:*

- $x = \rho \sin(\phi) \cos(\theta)$
- $y = \rho \sin(\phi) \sin(\theta)$
- $z = \rho \cos(\phi)$
- $\rho^2 = x^2 + y^2 + z^2$.

Next we need the integral formula. We again divide our region into wedges, but these are wedges of a spherical shell, rather than the blocks-of-cheese that feature in cylindrical coordinates.

Again the thickness is just $d\rho$. We need to compute the area of the inner square of the wedge. We see that the "height" is determined by the length of the ϕ arc, and thus is $\rho\, d\phi$.

The "width" is given by the length of the θ arc. In cylindrical coordinates this was given by $r\, d\theta$, but we'll have something smaller in spherical coordinates: as you move away from the $z = 0$ plane (which is also the $\phi = \pi/2$ plane!) the radius of the circle given by intersecting the plane $z = z_0$ with the sphere $\rho = \rho_0$ will decrease, proportionately to $\sin\phi$. Thus the height of the wedge is $\rho \sin\phi\, d\theta$, and our integral is

$$\int_{\rho_1}^{\rho_2} \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} f\left(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)\right) \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho.$$

Example 5.36. Let's find the volume of the unit sphere. We have

$$\begin{aligned}
\int_0^1 \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin(\phi)\, d\phi\, d\theta\, d\rho &= \int_0^1 \int_0^{2\pi} -\rho^2\cos(\phi)|_0^{\pi}\, d\theta\, d\rho \\
&= \int_0^1 \int_0^{2\pi} 2\rho^2\, d\theta\, d\rho \\
&= \int_0^1 4\pi\rho^2\, d\rho \\
&= \frac{4}{3}\pi\rho^3|_0^1 = 4\pi/3.
\end{aligned}$$

Example 5.37. Find the mass of a sphere with radius 3 and density equal to $\rho\cos^2\theta$.

$$\begin{aligned}
\int_0^3 \int_0^{2\pi} \int_0^{\pi} \rho^3 \cos^2\theta \sin\phi\, d\phi\, d\theta\, d\rho &= \int_0^3 \int_0^{2\pi} \rho^3\cos^2\theta(-\cos\phi)|0^{\pi}\, d\theta\, d\rho \\
&= \int_0^3 \int_0^{2\pi} 2\rho^3\cos^2\theta\, d\theta\, d\rho \\
&= \int_0^3 \rho^3 \left(\theta + \frac{\sin(2\theta)}{2}\ |)\right|_0^{2\pi} d\rho \\
&= \int_0^3 2\pi\rho^3\, d\rho = \pi\rho^4/2|_0^3 = \frac{81\pi}{2}.
\end{aligned}$$