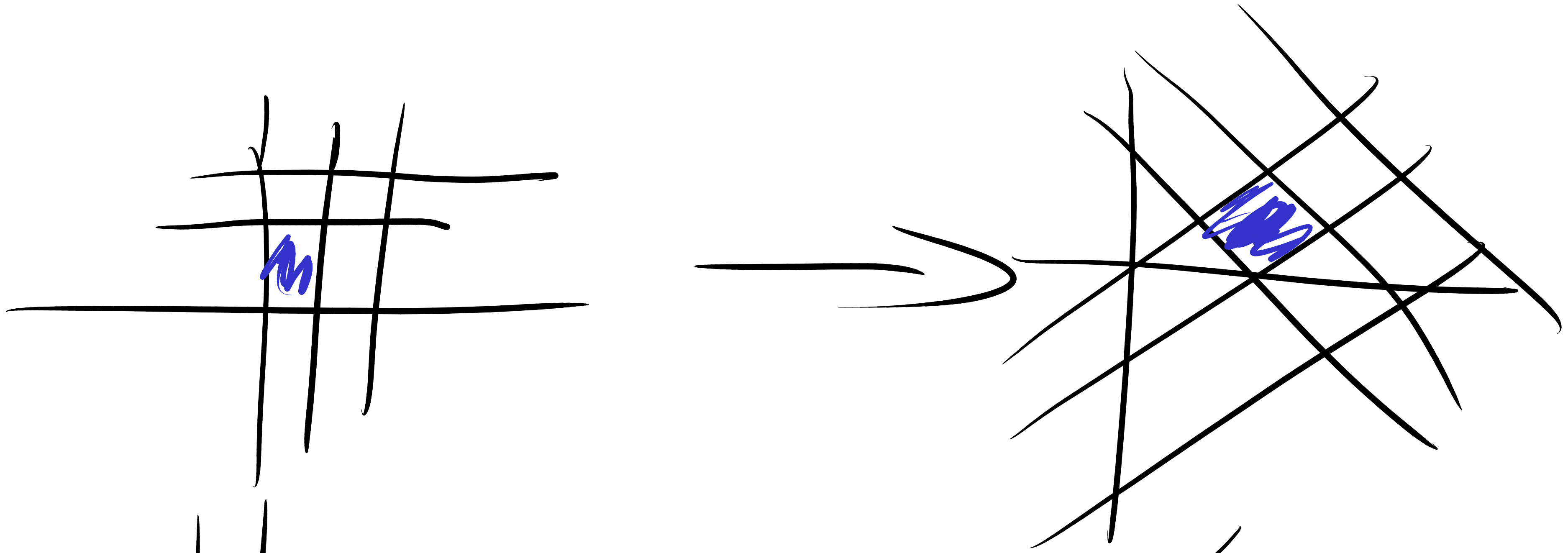
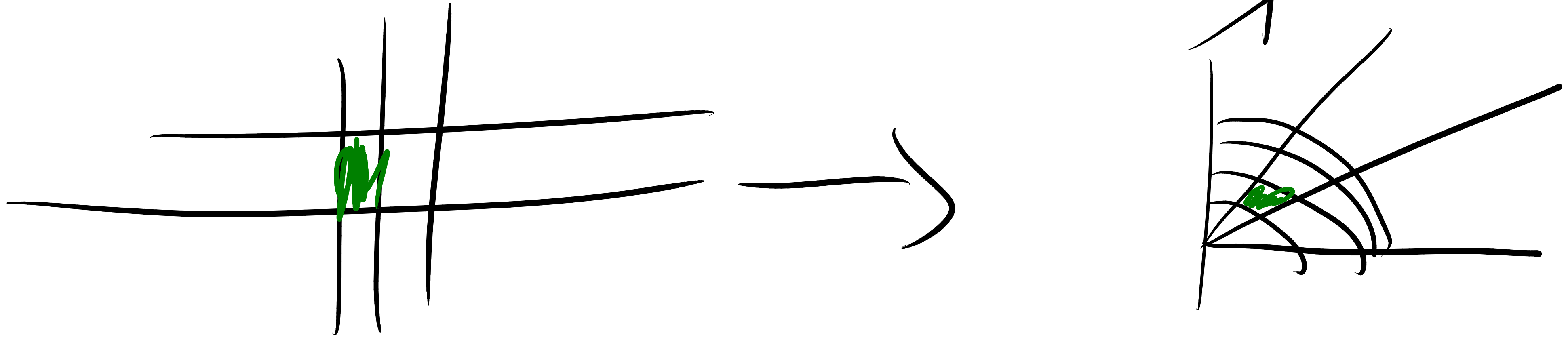
Change of coordinates

Jacobian: $\boxed{\dfrac{\partial(x,y)}{\partial(s,t)}} = \begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{vmatrix} = \dfrac{\partial x}{\partial s}\dfrac{\partial y}{\partial t} - \dfrac{\partial x}{\partial t}\dfrac{\partial y}{\partial s}$

$\int f(g(u)) \cdot \boxed{g'(u)}\, du$

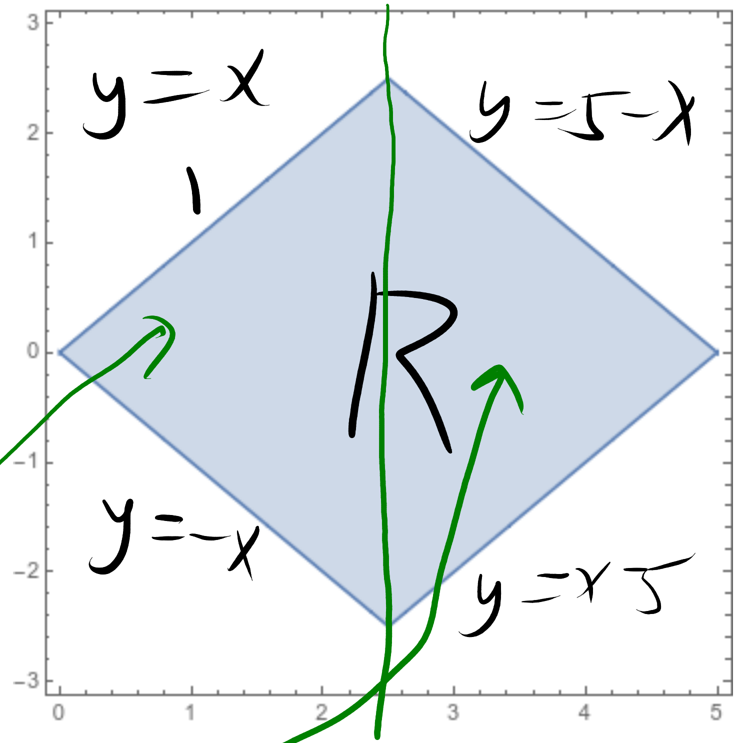


$r\, dr\, d\theta$



$$\int_R x+y \quad dA$$

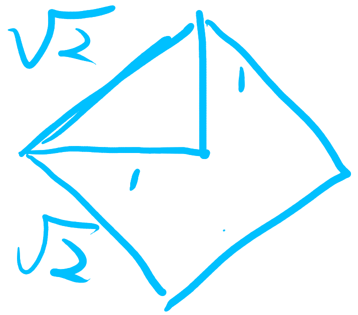
$(0,0)$, $(5,0)$, $(5/2, 5/2)$, $(5/2, -5/2)$



$$\int_0^{5/2}\int_{-x}^{x} x+y \, dy \, dx$$

$$\int_{5/2}^{5}\int_{x-5}^{5-x} x+y \, dy \, dx$$

$$x = s+t$$
$$y = s-t$$



$y=x \iff s+t = s-t \implies t=0$

$y=-x \iff s=0$

$y=5-x \iff s-t = 5-s-t \implies s = \frac{5}{2}$

$y = x-5 \iff t = 5/2$

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2$$

$$\int_0^{5/2}\int_0^{5/2} (s+t)+(s-t) \left|\frac{\partial(x,y)}{\partial(s,t)}\right| ds \, dt = \int_0^{5/2}\int_0^{5/2} 2s(2) \, ds \, dt$$

§6.4 Vector Fields

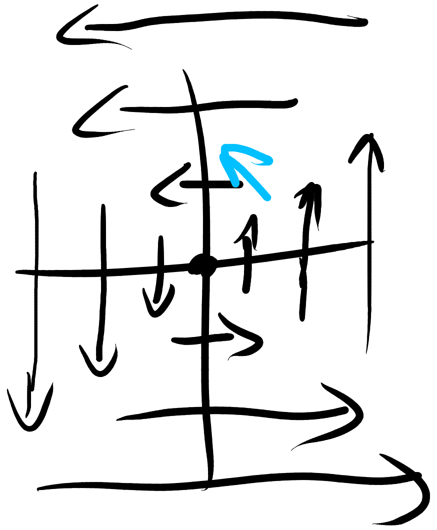
$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ $(\mathbb{R}^2 \to \mathbb{R}^2)$

point $\to$ vector

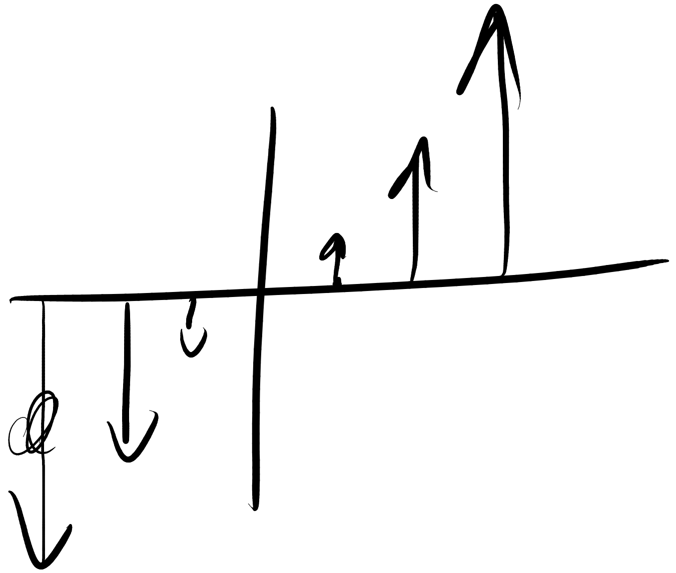
Dfn: a vector field in $\mathbb{R}^n$ is a fn $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ that takes in a pt and outputs a vector.

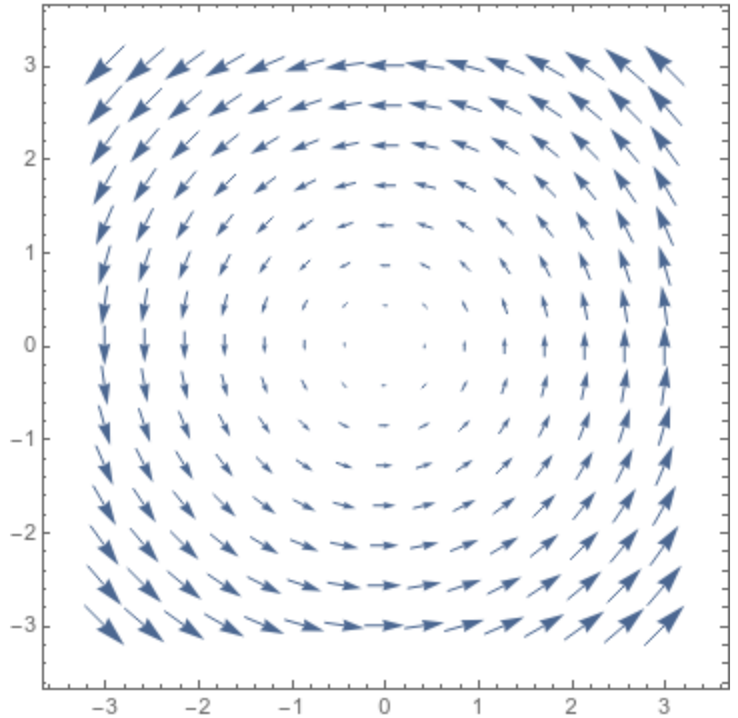
$\vec{F}(x,y) = -y\vec{i} + x\vec{j}$

$\vec{F}(x,0) = x\vec{j}$ $\quad$ $\vec{F}(0,y) = -y\vec{i}$

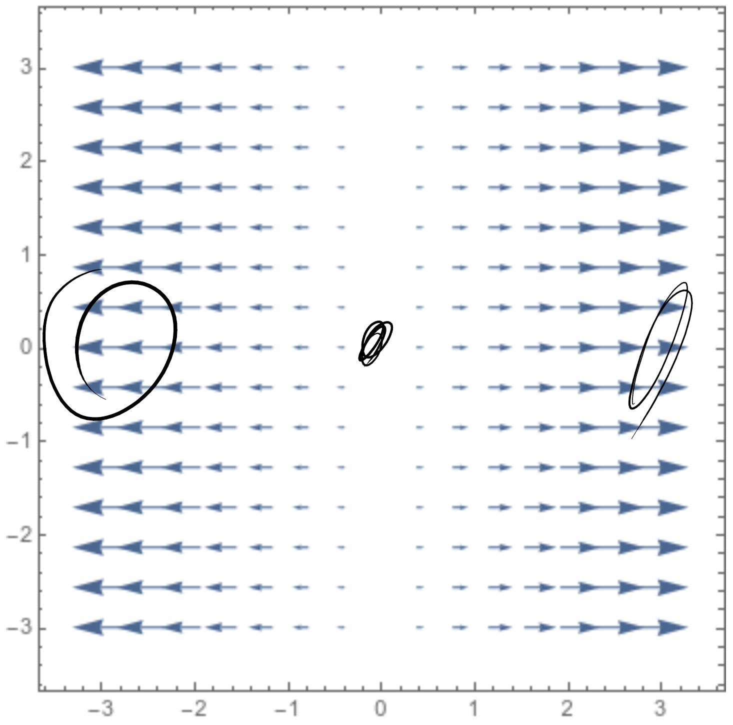


$\vec{F}(1,1) = -\vec{i} + \vec{j}$

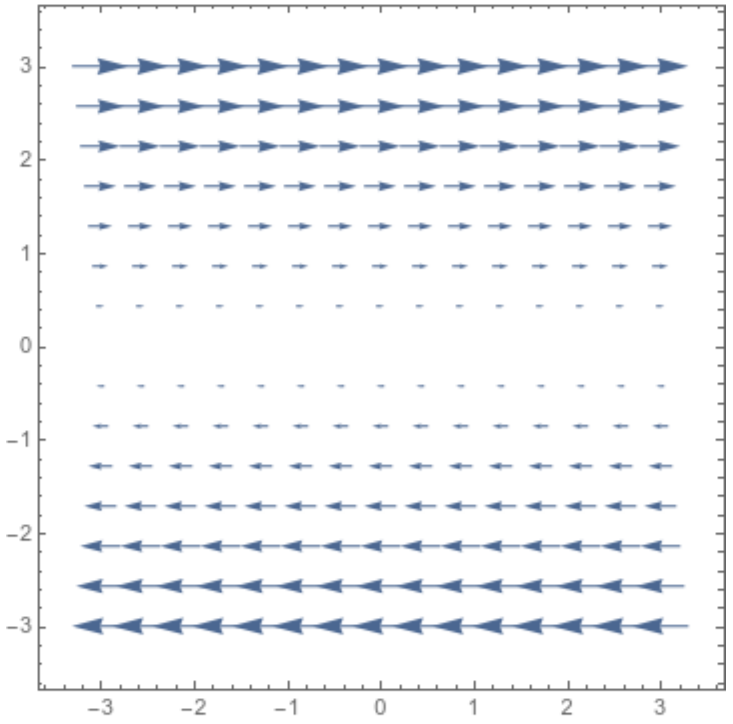




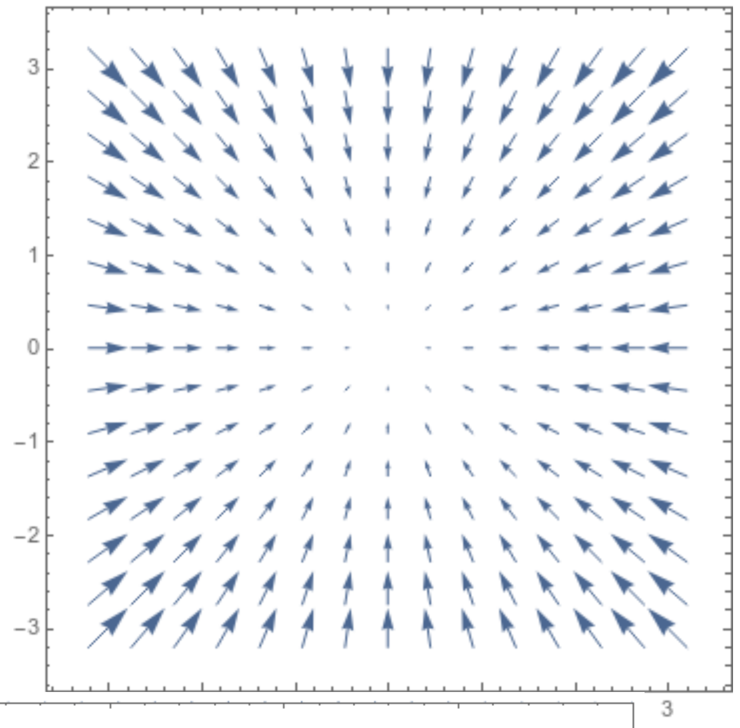
$\vec{F}(x,y) = x\vec{i}$



$\vec{F}(x,y) = y\vec{i}$

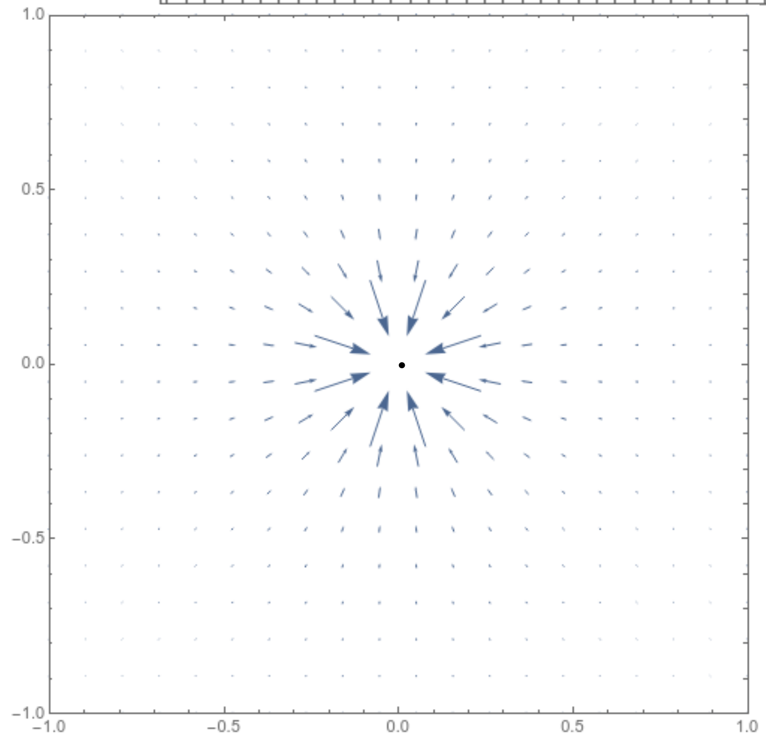


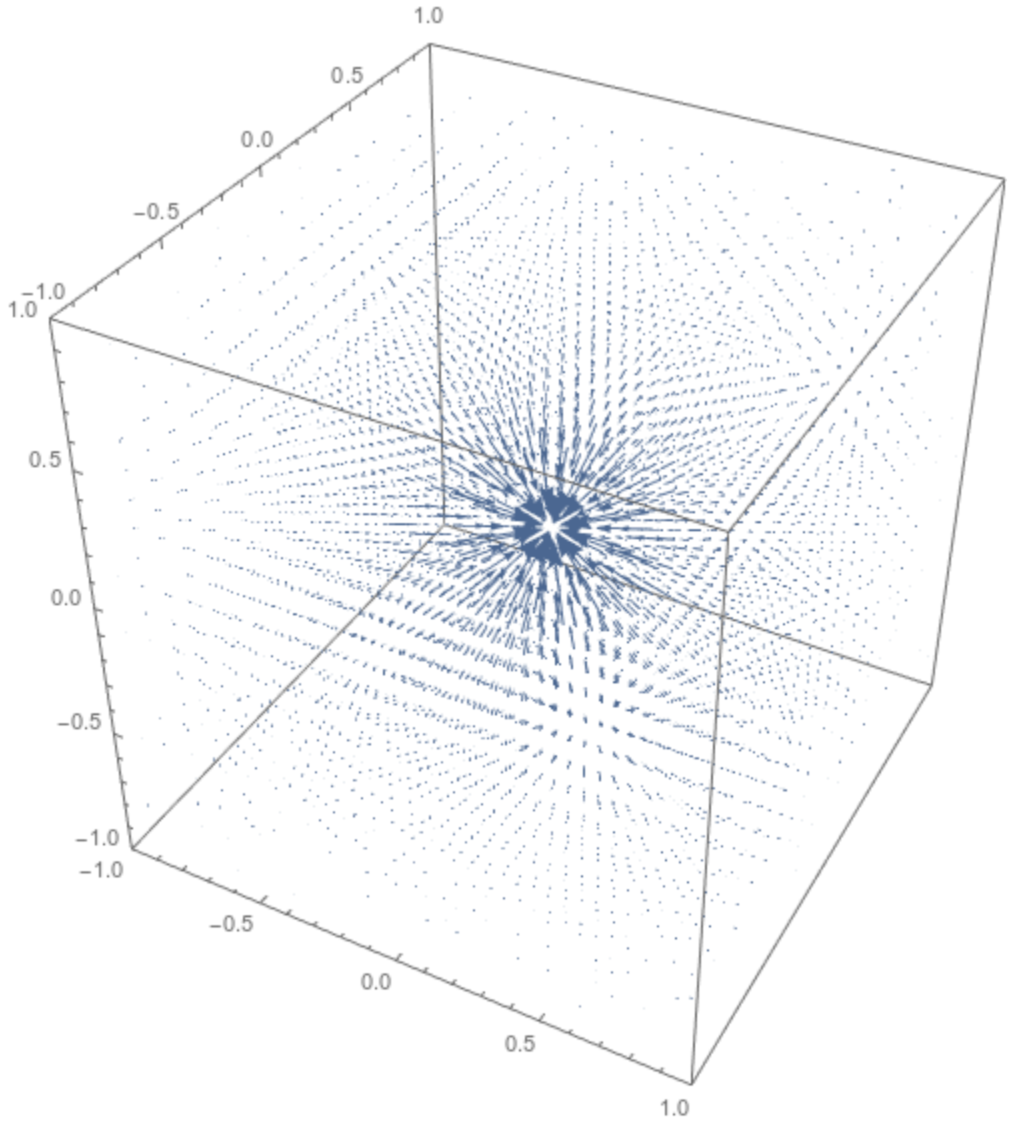
$\vec{F}(x,y) = -x\vec{i} - y\vec{j}$



Gravity

$$\vec{F}(x,y) = \frac{-x\vec{i} - y\vec{j}}{\left(\sqrt{x^2+y^2}\right)^3}$$

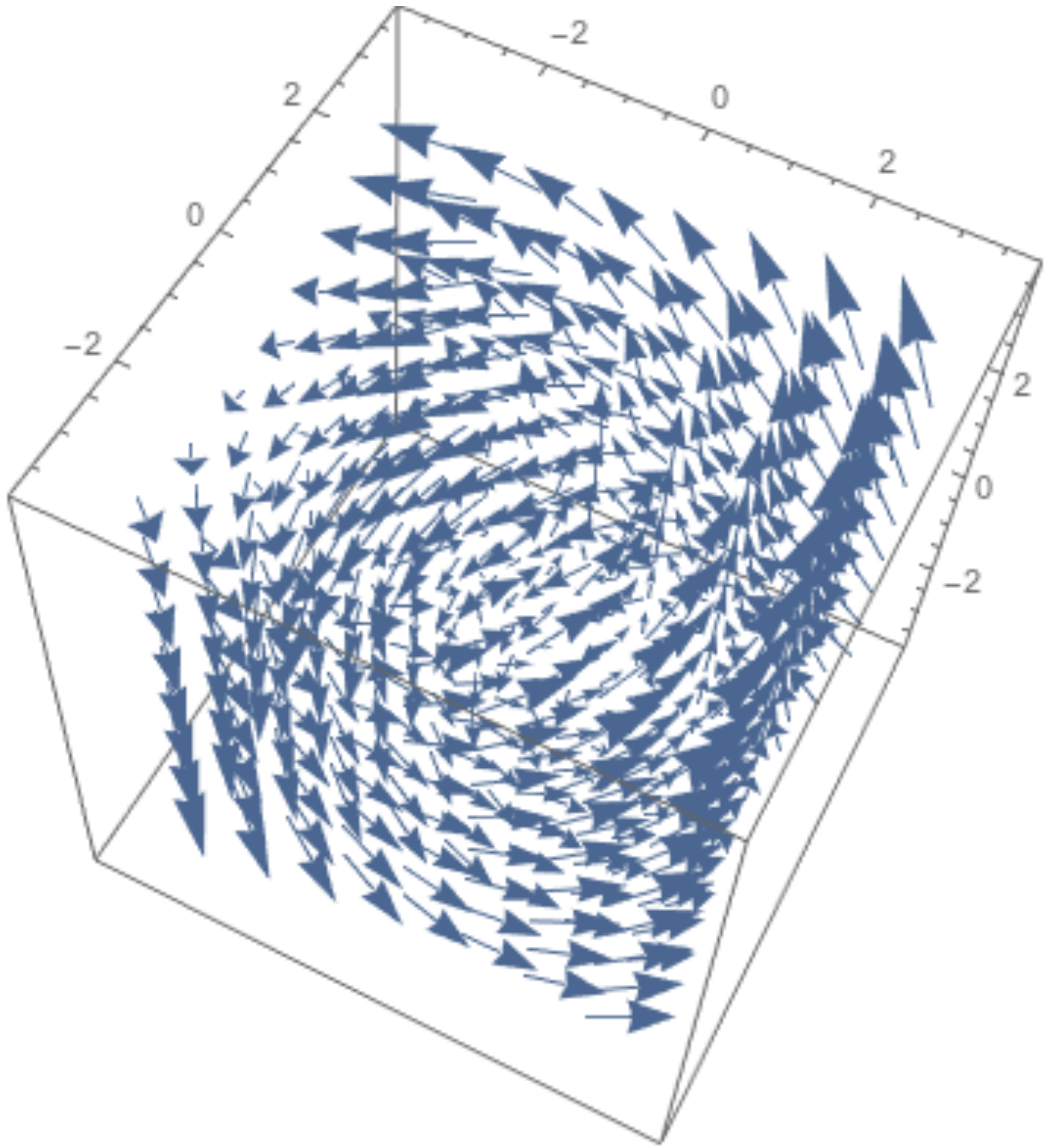


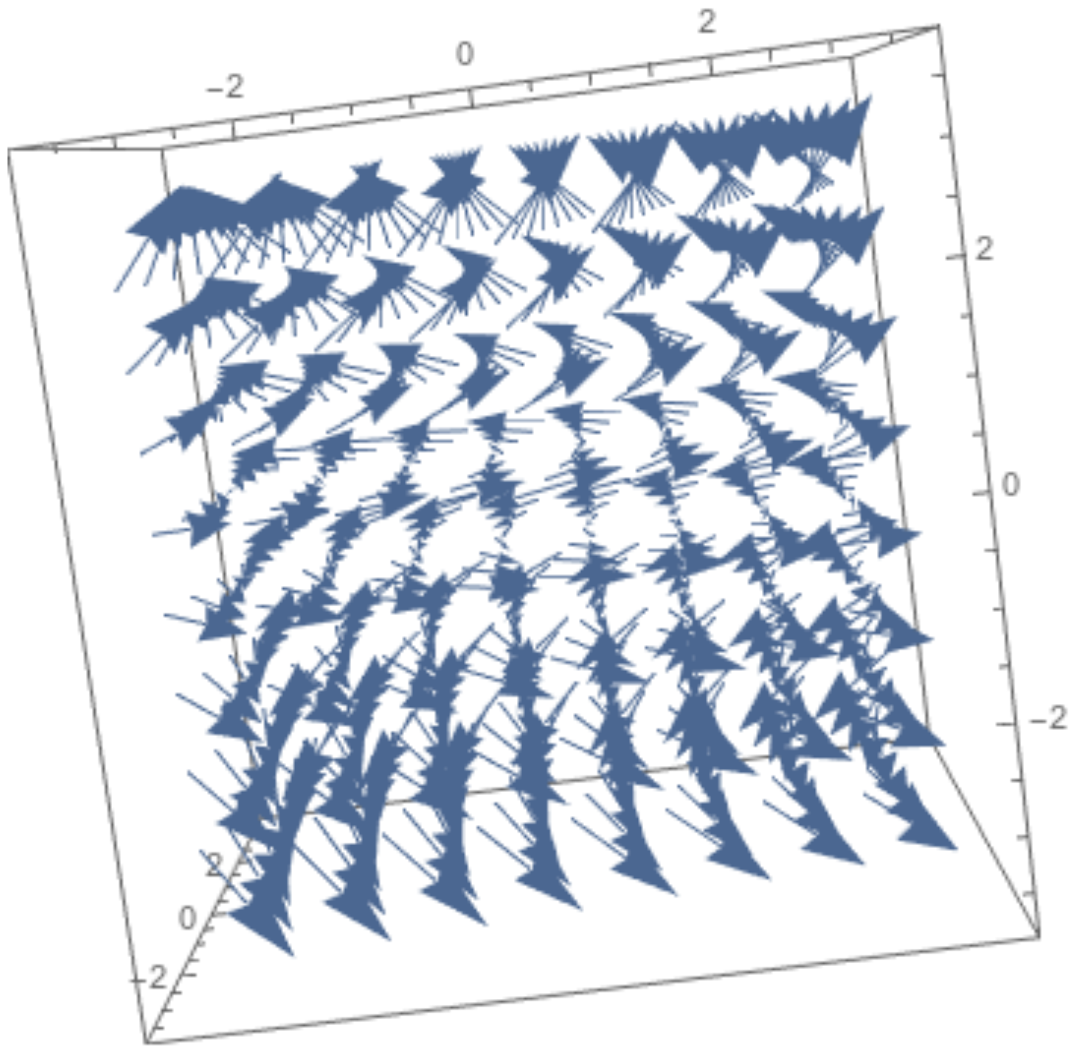


$$\vec{F}(\vec{r}) = \frac{GMm}{\|\vec{r}\|^2} \frac{-\vec{r}}{\|\vec{r}\|}$$

$$= \frac{GMm}{\|\vec{r}\|^3} (-\vec{r})$$

$$\vec{F}(x,y,z) = -y\,\vec{i} + x\,\vec{j} + z\,\vec{k}$$

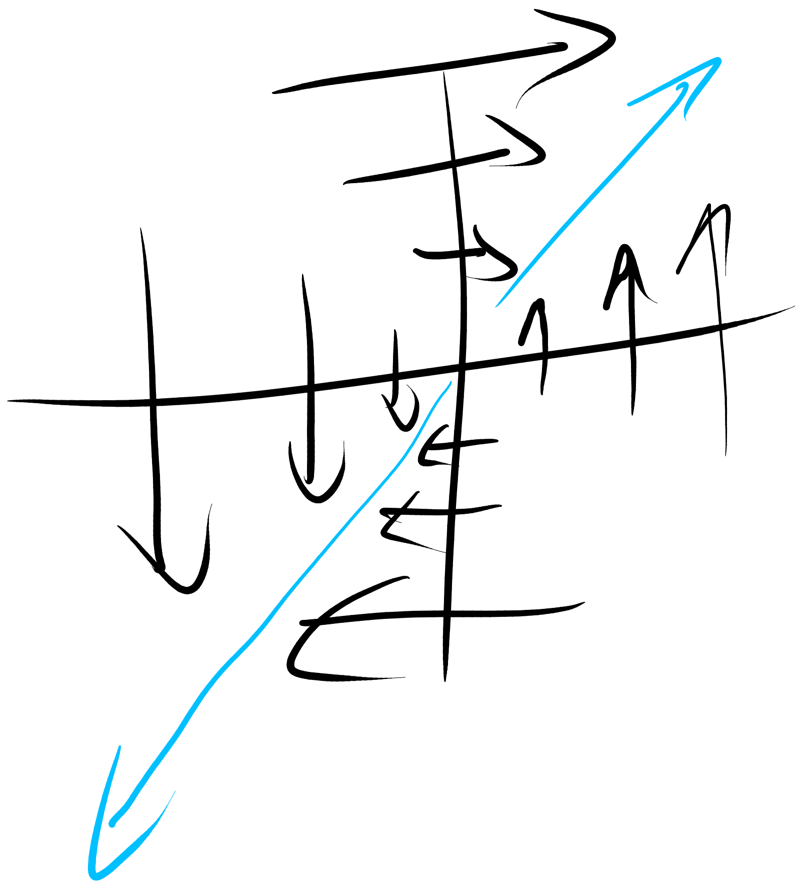


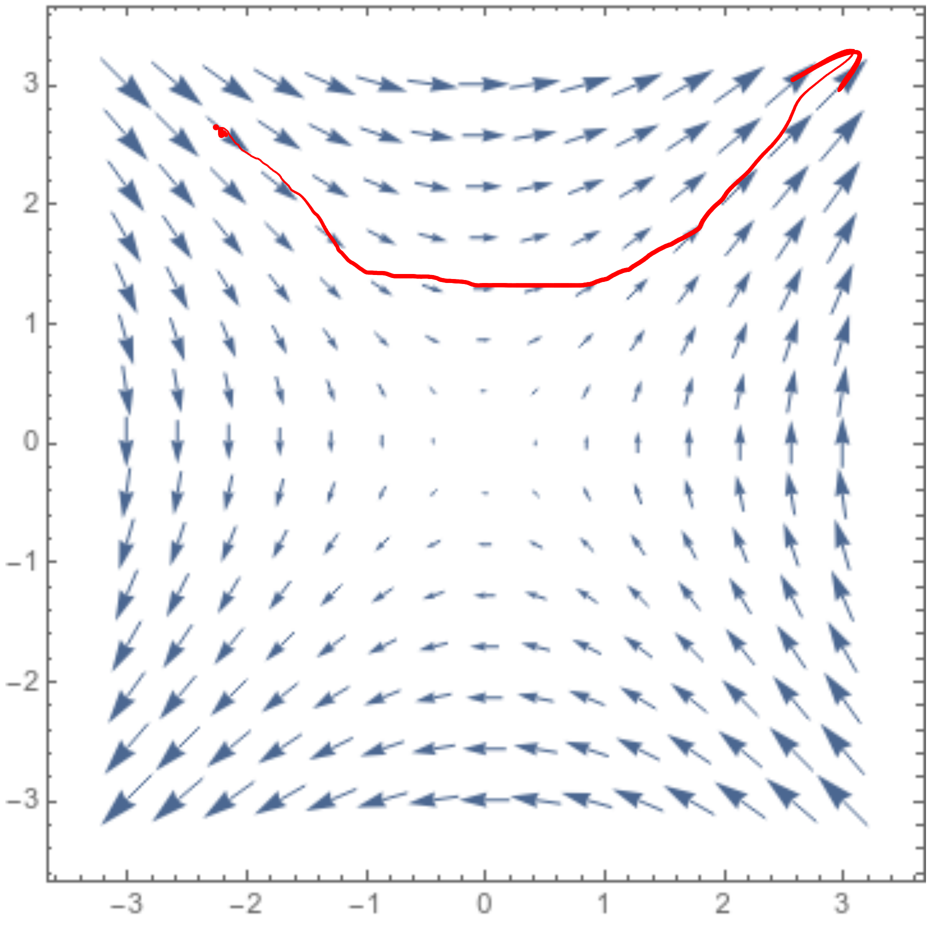


Important source of vector fields:

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

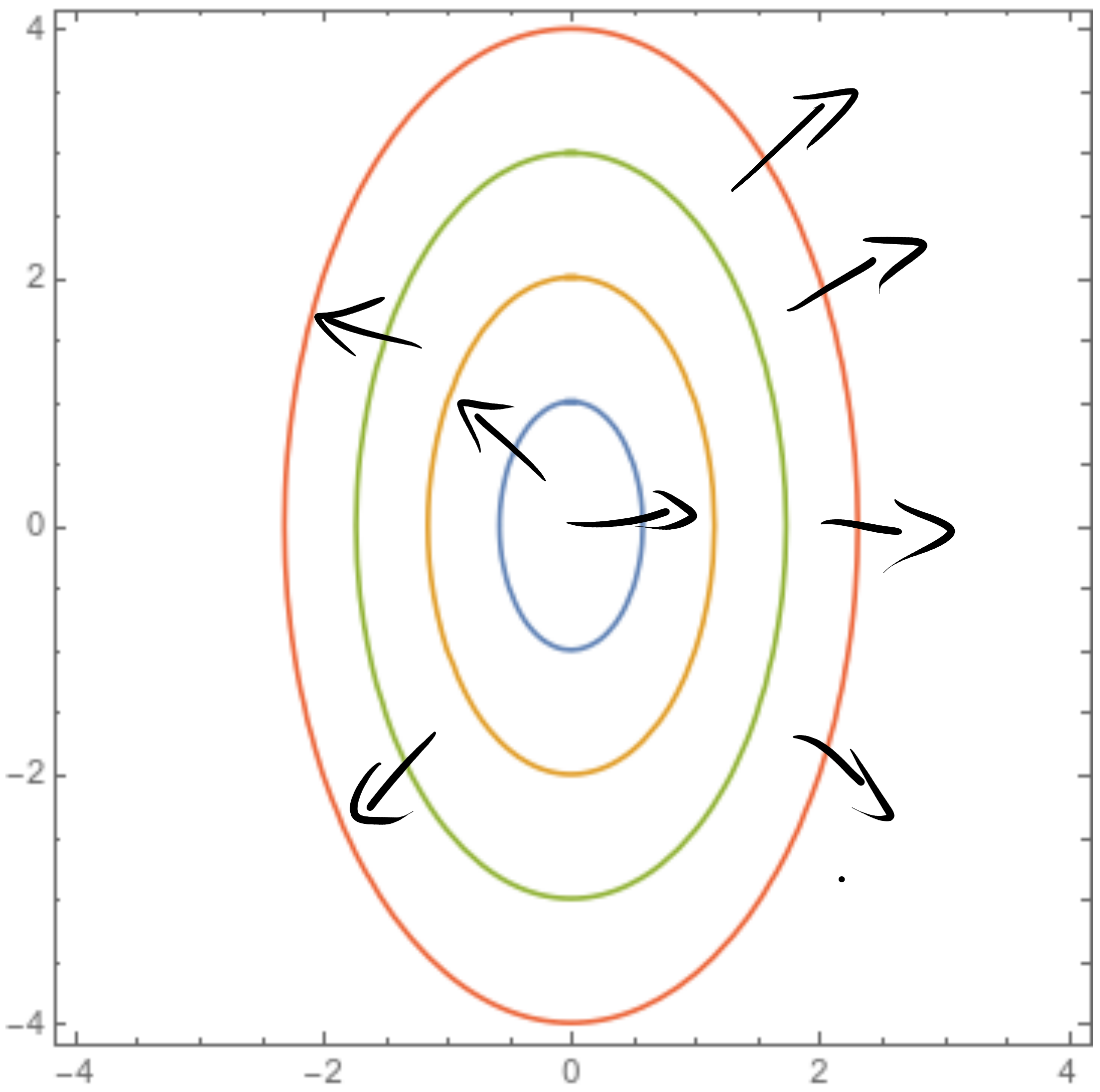
$f(x,y) = xy \qquad \nabla f(x,y) = (y, x)$

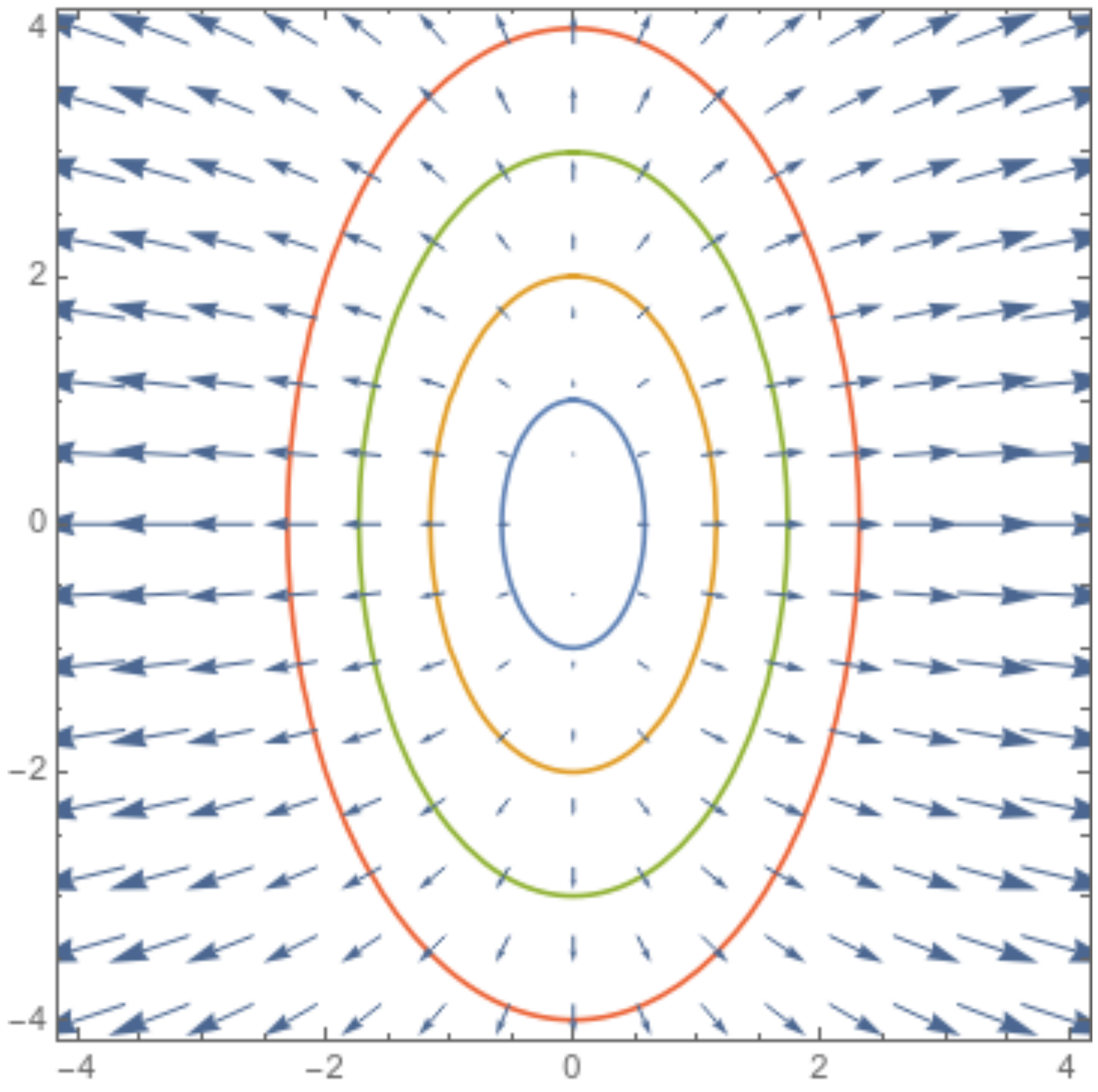




Grad v F is $\perp$ to level sets

$f(x,y) = 3x^2 + y^2$





Dfn: $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ VF

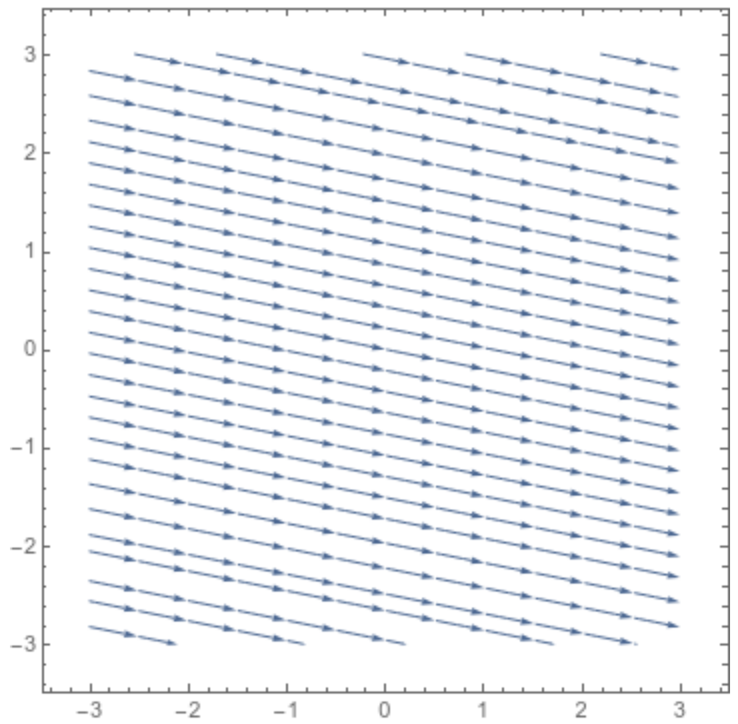
$\vec{r}: \mathbb{R} \to \mathbb{R}^n$

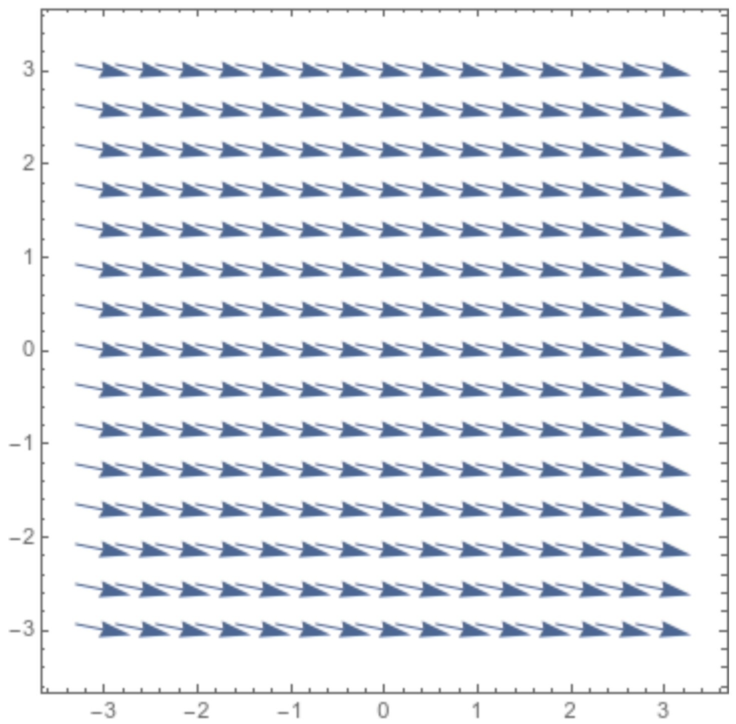
$\vec{r}$ is a <u>flowline</u> of $\vec{F}$

if $\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$

The <u>flow</u> of a VF is the collection of all flow lines.

$\vec{v}(t) = 5\vec{i} - \vec{j}$





flowline through $(3,3)$

$$\vec{r}(t) = (3,3) + (\vec{v}(3,3))t$$
$$= (3+5t,\ 3-t).$$

$\vec{F}(x,y) = y\vec{i} + \vec{j}$

find path of object at $(2,2)$ at $t=0$

$\vec{r}(0) = (2,2)$

$\vec{r}'(t) = (y,1)$

$= (y(t),1)$

$\vec{r}'(t) = (x'(t), y'(t))$

$x'(t) = y(t)$

$y'(t) = 1$

$y(t) = t + y_0$

$y(t) = t+2$

$x'(t) = t+2$

$x(t) = \frac{t^2}{2} + 2t + x_0$

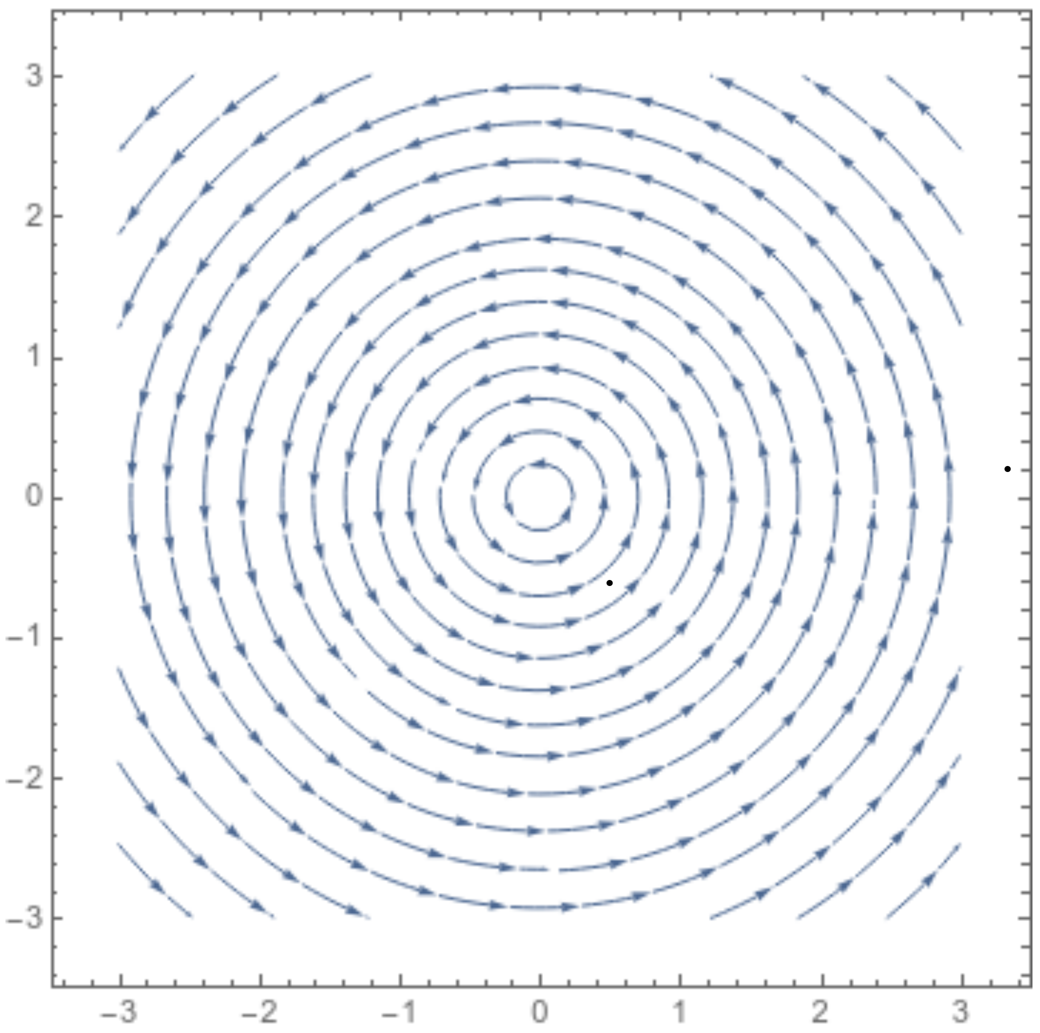
$= \frac{t^2}{2} + 2t + 2$

$\vec{r}(t) = \left(\frac{t^2}{2} + 2t + 2,\ t+2\right).$

$$\vec{F}(x,y) = -y\vec{i} + x\vec{j}$$

$$x'(t) = -y(t)$$
$$y'(t) = x(t)$$

$$x''(t) = -x(t)$$
$$y''(t) = -y(t)$$



Know: $x(t) = A\cos t \cancel{+ B\sin t}$

$$y(t) = A\sin t$$

$$\vec{r}(t) = (A\cos t, A\sin t)$$