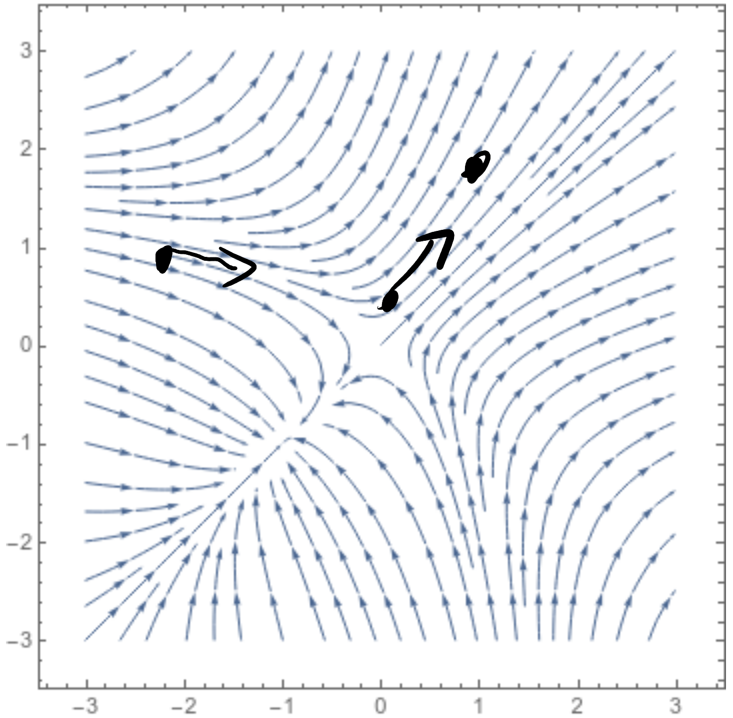
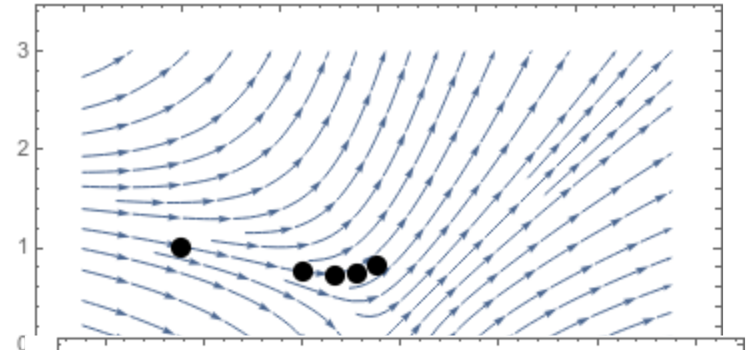
$\vec{F}(x,y) = (x^2+y)\,\vec{i} + (x+y^2)\,\vec{j}$. $\vec{r}$ is a flow line, $\vec{r}(0) = (-2,1)$. $\vec{r}(1) = ?$



$\vec{r}\,'(0) = \vec{F}(-2,1) = 5\vec{i} - \vec{j}$

$\vec{r}(1) \approx \vec{r}(0) + \vec{r}\,'(0)(1-0) = (-2,1) + (5,-1) = 3, 0$

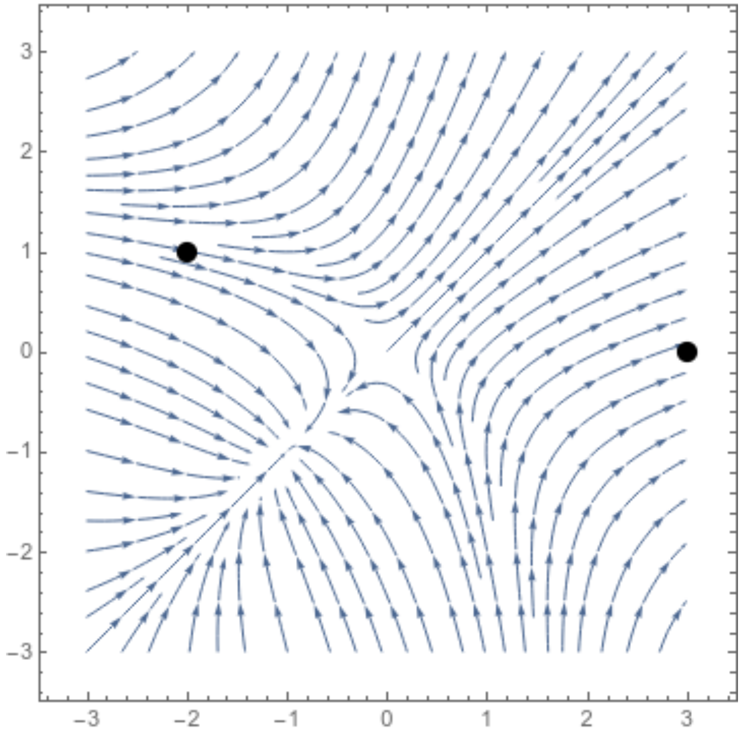


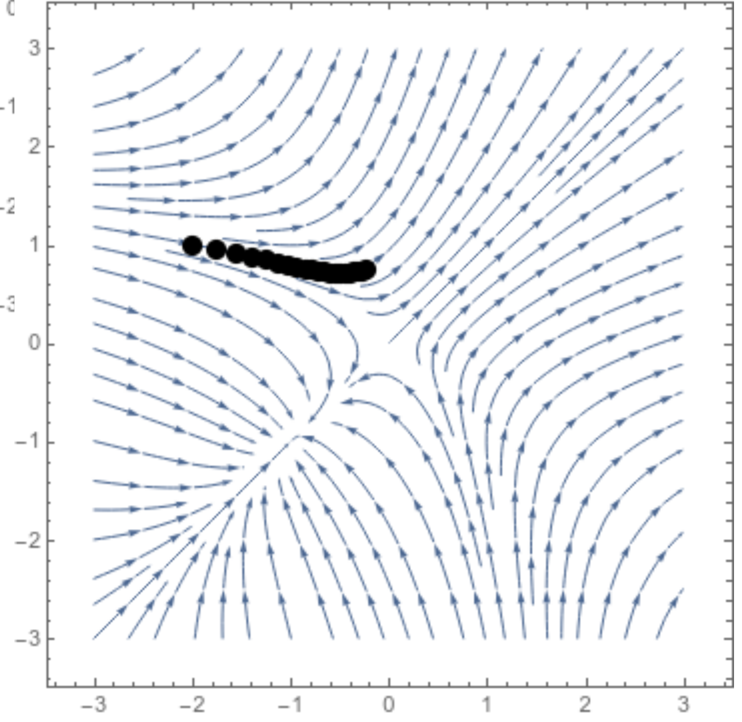
$\vec{r}\,'(0) = 5\vec{i} - \vec{j}$

$\vec{r}(1/2) \approx \vec{r}(0) + \vec{r}\,'(0)(1/2) = (1/2, 1/2)$

$\vec{r}(1) \approx \vec{r}(1/2) + \vec{r}\,'(1/2)(1-1/2)$

$\approx (1/2, 1/2) + \vec{F}(1/2, 1/2)(1/2) = (7/8, 7/8)$





§7 Line Integrals

1) Lots of types of integrals
2) Ways to not do them.

§7.1 Integrating over a curve.

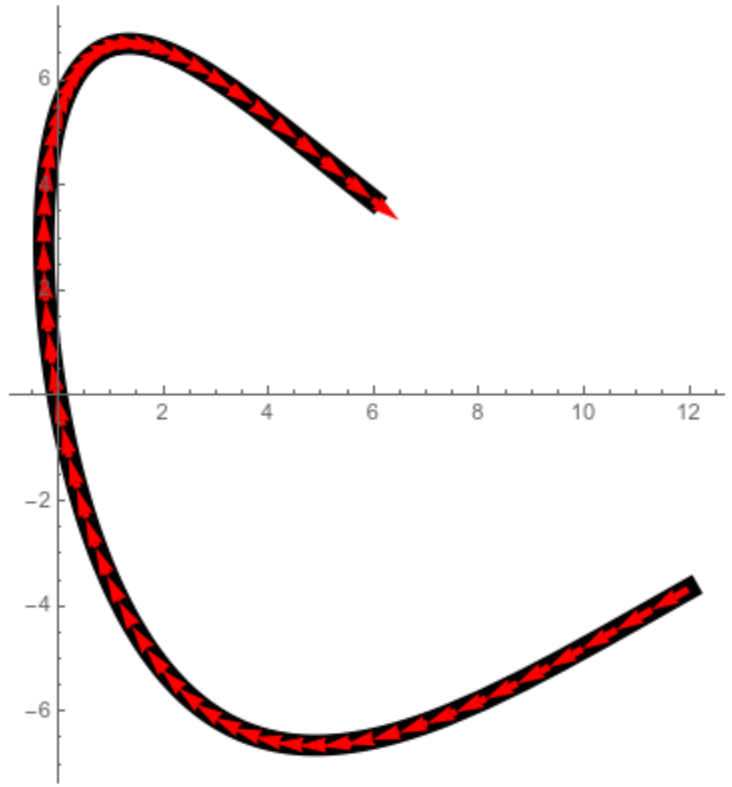
Integrate a multivariable fn over a one-dimensional curve.

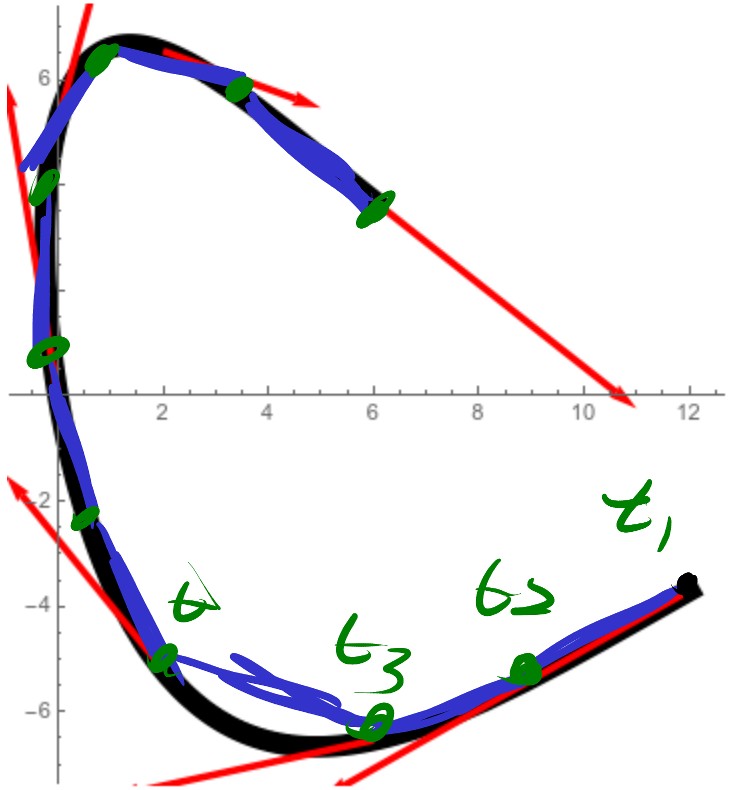
Ex: Arc Length

$\vec{r}(t) = (x(t), y(t), z(t))$

$$L \approx \sum_{i=1}^{n} \left\| \vec{r}(t_{i+1}) - \vec{r}(t_i) \right\|$$

$$= \sum_{i=1}^{n} \left\| \frac{\vec{r}(t_{i+1}) - \vec{r}(t_i)}{\Delta t} \right\| \Delta t$$





$$L \approx \sum_{i=1}^{n} \left\| \vec{r}(t_{i+1}) - \vec{r}(t_i) \right\|$$

$$= \sum_{i=1}^{n} \left\| \frac{\vec{r}(t_{i+1}) - \vec{r}(t_i)}{\Delta t} \right\| \Delta t$$

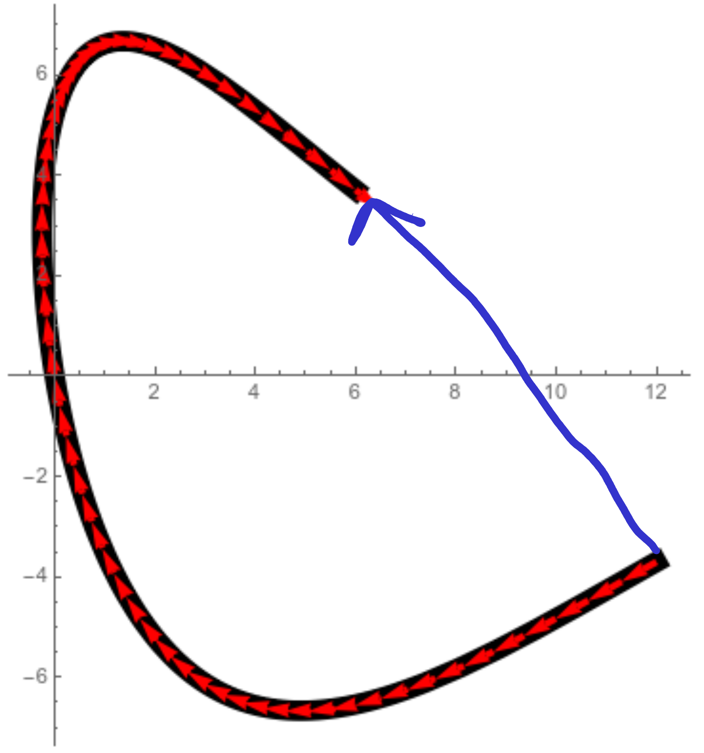
$$\approx \sum_{i=1}^{n} \left\| \vec{r}\,'(t_i) \right\| \Delta t$$

$$\approx \int_{t_0}^{t_f} \|\vec{r}\,'(t)\| \, dt$$

displacement

$$\int \vec{r}\,'(t)\, dt$$

$$= r(t_f) - r(t_0)$$



$$\int \text{speed} \cdot d\,\text{time} = \text{distance}$$

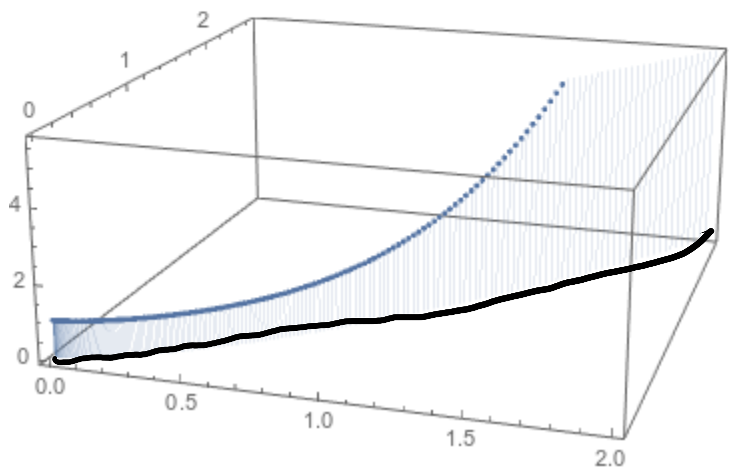
$C: \vec{r}(t) = (2\cos(t), \sin(t)), \quad 0 \le t \le 2\pi$

$$\int_0^{2\pi} \|\vec{r}\,'(t)\|\,dt = \int_0^{2\pi} \|(-2\sin(t), \cos(t))\|\,dt$$

$$= \int_0^{2\pi} \sqrt{4\sin^2 t + \cos^2 t}\,dt \approx 9.69$$

"elliptic integral"

Generalize arc integrals



$$I \approx \sum_{i=1}^{n} f(\vec{r}(t_i)) \, \|\vec{r}(t_{i+1}) - \vec{r}(t_i)\|$$

$$\approx \sum_{i=1}^{n} f(\vec{r}(t_i)) \, \|\vec{r}\,'(t_i)\| \, \Delta t_i$$

$$\approx \int_{t_0}^{t_f} \underbrace{f(\vec{r}(t))}_{\text{fuel burned to move that distance}} \underbrace{\|\vec{r}\,'(t)\| \, dt}_{\text{distance}}$$

Dfn: The <u>scalar line integral</u> of $f(\vec{r})$ over C is

$$\int_C f(\vec{r}) \, ds = \int_a^b f(\vec{r}(t)) \, \|\vec{r}\,'(t)\| \, dt$$

Ex: C: Line from $(1,1)$ to $(3,5)$ $\vec{r}(t) = (1+2t, 1+4t)$

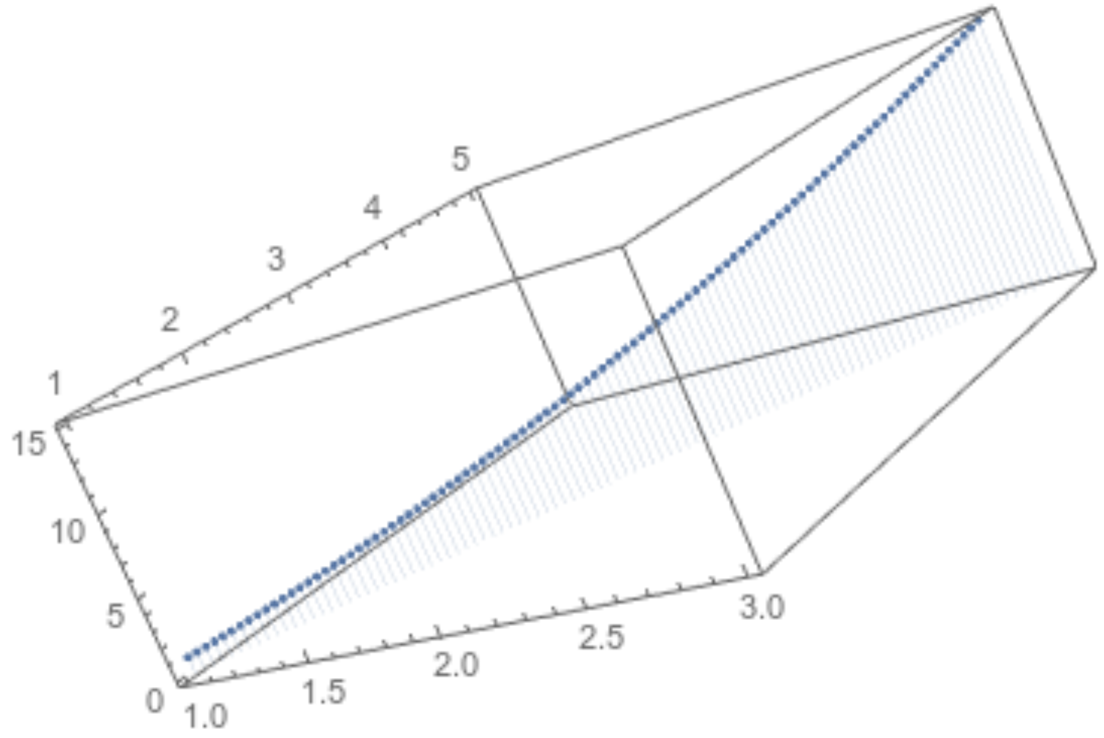
$0 \le t \le 1$

$$\int_C xy\, ds = \int_0^1 x(t)\, y(t)\, \|\vec{r}\,'(t)\|\, dt$$

$$= \int_0^1 (1+2t)(1+4t)\, \|(2,4)\|\, dt$$

$$= 2\sqrt{5} \int 1 + 6t + 8t^2\, dt$$

$$= \frac{40\sqrt{5}}{3}$$



EX: helical spring

lies along $\bar{r}(t) = (\cos(t), \sin(t), t)$, $0 \le t \le 2\pi$

density is $1+z$

find the mass

$$\int_C 1+z \, ds = \int_0^{2\pi} (1+z(t)) \, \|\bar{r}'(t)\| \, dt$$

$$= \int_0^{2\pi} (1+t) \, \|(-\sin(t), \cos(t), 1)\| \, dt$$

$$= \int_0^{2\pi} (1+t) \sqrt{\sin^2 t + \cos^2 t + 1} \, dt = \int_0^{2\pi} (1+t) \sqrt{2} \, dt = 2\pi\sqrt{2}(1+\pi).$$

follow a path $\bar{r}(t) = \left(\frac{t^2}{2}, \frac{t^3}{3}\right)$ for $0 \le t \le 2$

use fuel $1 + xy^2 / m$ total fuel used?

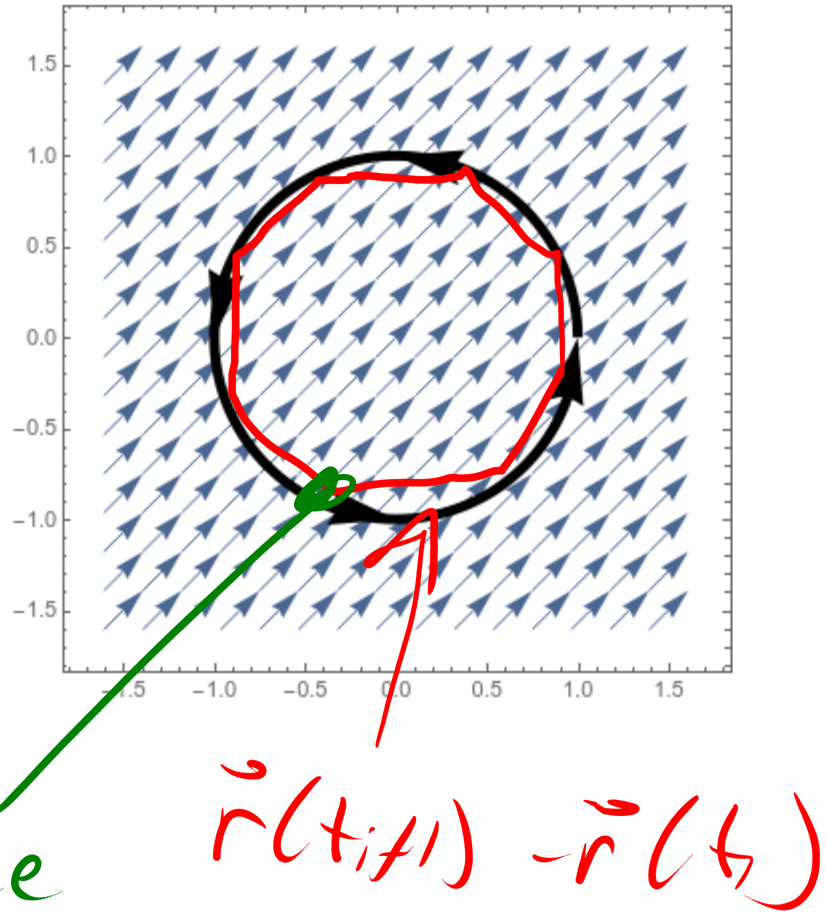
$$\int_C (1+xy^2)\, ds = \int_0^2 \left(1 + \frac{t^2}{2}\left(\frac{t^3}{3}\right)^2\right) \|(t, t^2)\|\, dt$$

$$= \int_0^2 \left(1 + \frac{t^8}{18}\right)\sqrt{t^2+t^4}\, dt = \int_0^2 t\sqrt{1+t^2} + \frac{t^9}{18}\sqrt{1+t^2}\, dt \simeq 15.2$$

§7.2 Integrating VFs over a curve

Work is $\vec{F} \cdot \vec{d}$

$$\int \vec{F}(\vec{r}) \cdot d\vec{s}$$



Adding up $\vec{F}(\vec{s}) \cdot d\vec{s}$

$$W_i = \vec{F}_i \cdot d\vec{s}_i$$

$$= \vec{F}(\vec{r}(t_i)) \cdot (\vec{r}(t_{i+1}) - \vec{r}(t_i))$$

$$W \approx \sum_{i=1}^{n} \vec{F}(\vec{r}(t_i)) \cdot (\vec{r}(t_{i+1}) - \vec{r}(t_i)) \simeq \sum_{i=1}^{n} \vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i)\,\Delta t \approx \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

Dfn: The <u>line integral</u> of $\vec{F}$ over $C: \vec{r}(t)$ is

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{\|\Delta \vec{r}_i\| \to 0} \sum_{i=1}^{n} \vec{F}(\vec{r}_i) \cdot \Delta \vec{r}_i$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt.$$