Integrating Vector fields over a curve

$$\int_C \vec{F}\cdot d\vec{r} = \lim_{\|\Delta \vec{r}_i\|\to 0} \sum_{i=1}^{n} \vec{F}(\vec{r}_i)\cdot \Delta\vec{r}_i = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

- Work done by force field $\vec{F}$ to a particle moving along $C$
- Circulation along a path

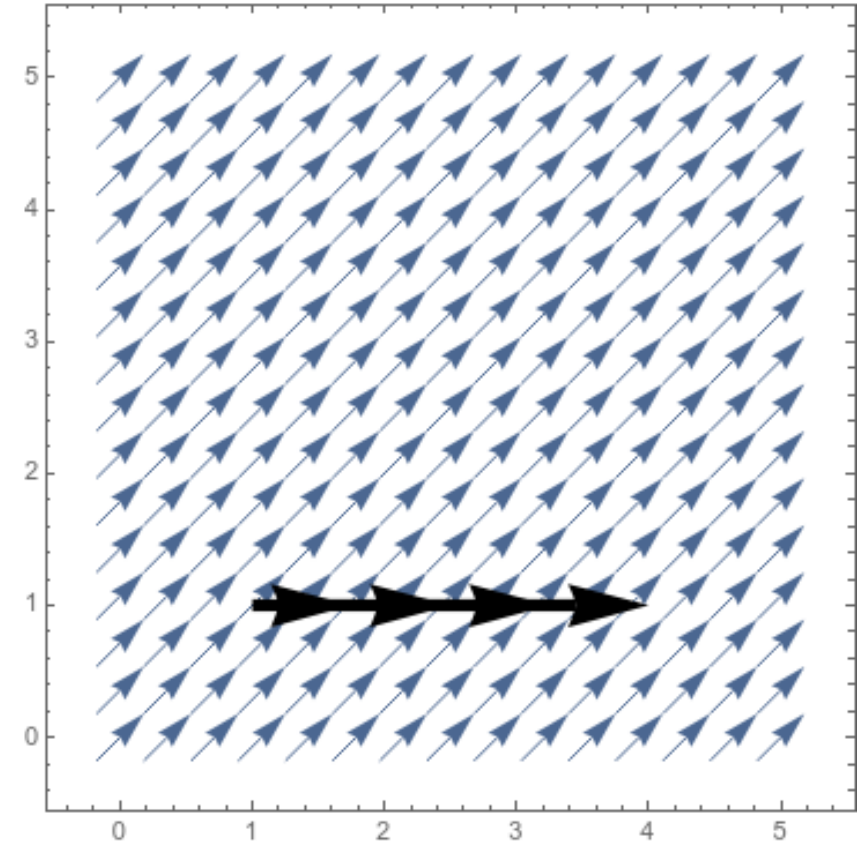
Ex: $\vec{F}(x,y) = \vec{i} + \vec{j}$, $C$ line from $(1,1)$ to $(4,1)$

$W = \vec{F}\cdot\vec{d} = (1,1)\cdot(3,0) = 3$

$\vec{r}(t) = (1+3t, 1) \quad 0 \le t \le 1$

$$\int_0^1 \vec{F}(1+3t,1)\cdot(3,0)\,dt$$

$$= \int_0^1 (1,1)\cdot(3,0)\,dt = 3.$$



$C$ circle centered at origin ↺ starting from $(1,0)$

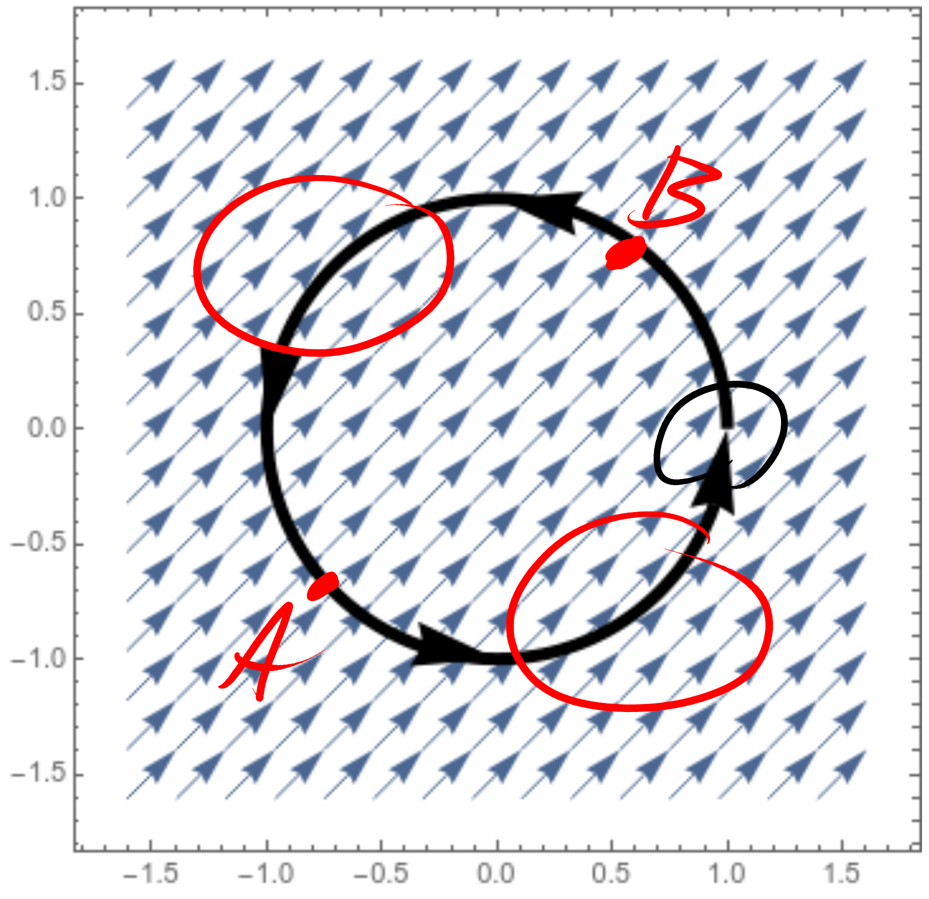
$\vec{r}(t) = (\cos(t), \sin(t))$ $\quad \vec{F}(x,y) = \vec{i} + \vec{j}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\cos(t), \sin(t)) \cdot (\cos t, \sin t)'\, dt$$

$$= \int_0^{2\pi} (1,1)\cdot(-\sin t, \cos t)\, dt$$

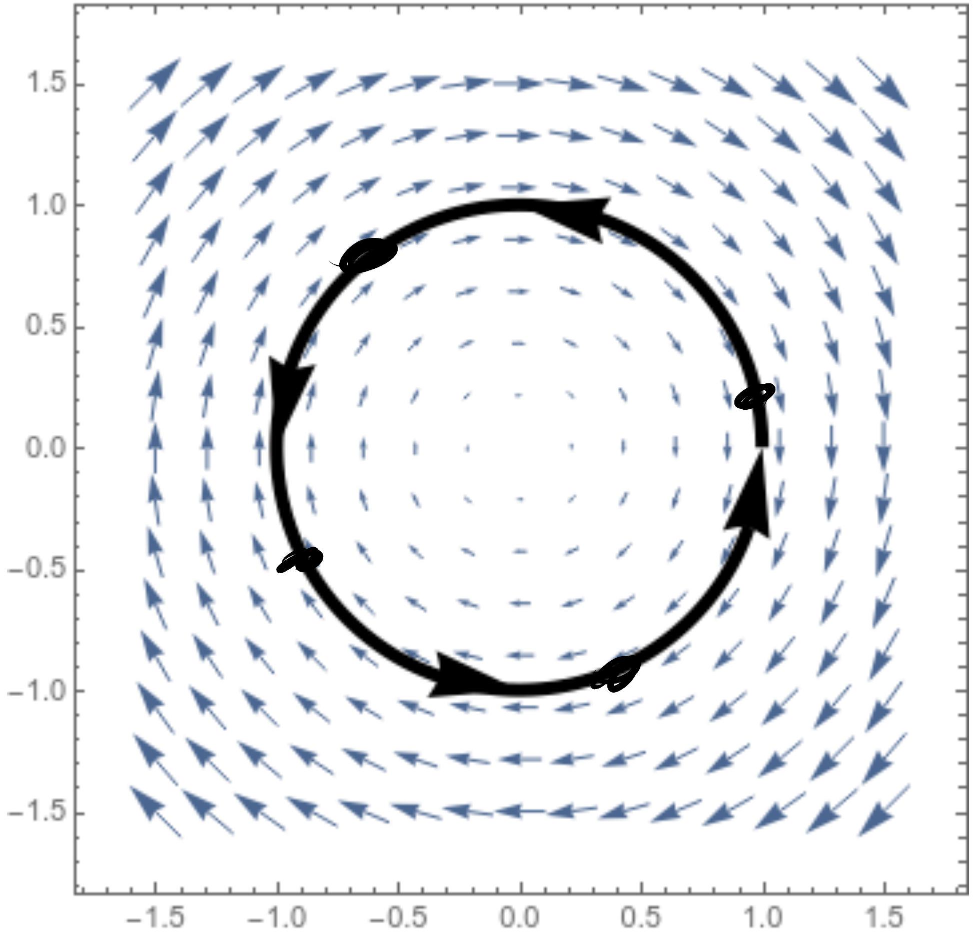
$$= \int_0^{2\pi} \cos t - \sin t \, dt = \sin t + \cos t \Big|_0^{2\pi}$$

$$= 0 - 0 + 1 - 1 = 0$$



Ex! $\vec{F}(x,y) = (y, -x) = y\vec{i} - x\vec{j}$

$\vec{r}(t) = (\cos(t), \sin(t))$




$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot r'(t)\, dt$$

$$= \int_0^{2\pi} (\sin t\, \vec{i} - \cos t\, \vec{j}) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} -\sin^2 t - \cos^2 t\, dt = \int_0^{2\pi} -1\, dt = -2\pi.$$

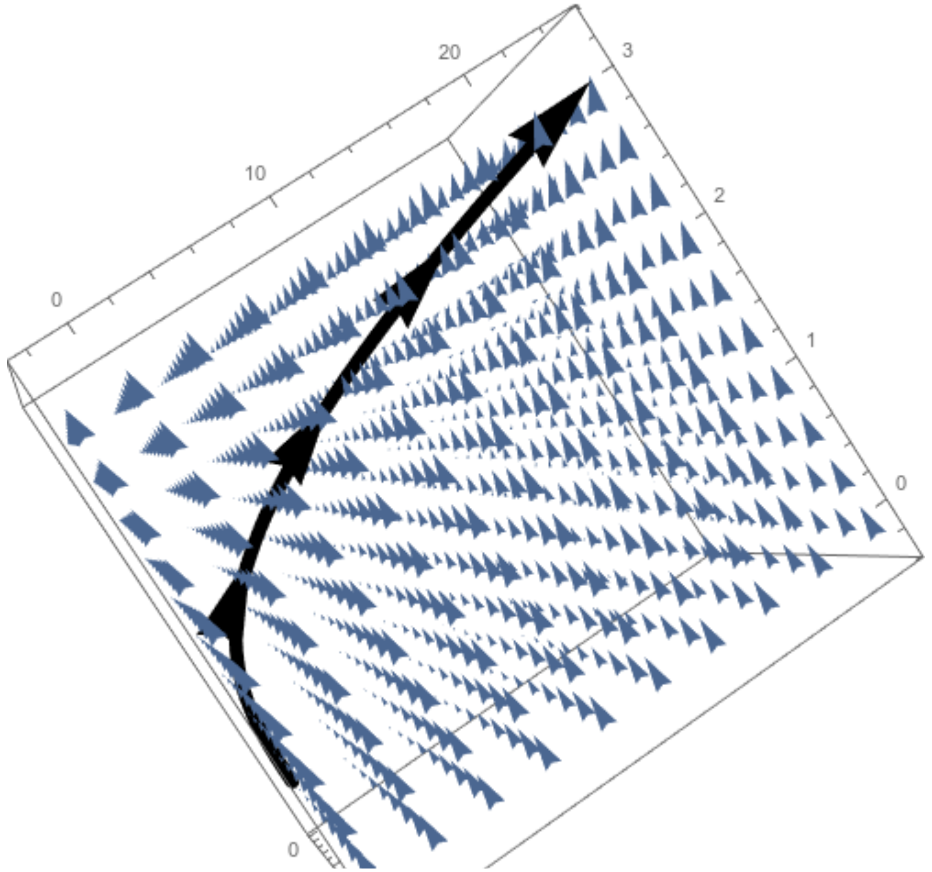
Ex: $\vec{r}(t) = (t, 3t^2, t^3 - 2) \quad 0 \le t \le 3$

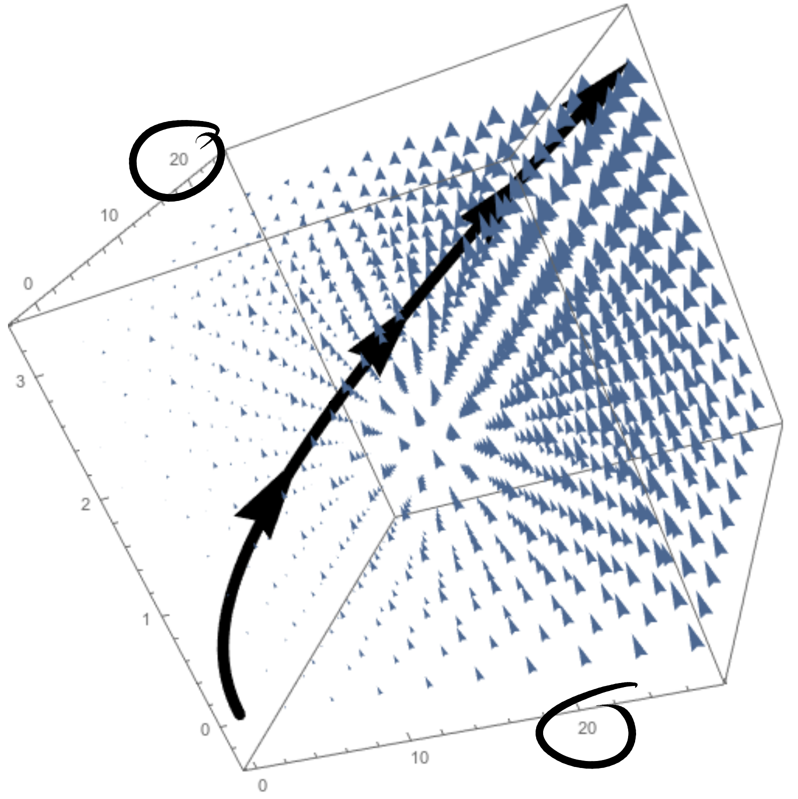
$\vec{F}(x,y,z) = y\,\vec{i} + x^2\,\vec{j} + xz\,\vec{k}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^3 (3t^2, t^2, t^4 - 2t)\cdot(1, 6t, 3t^2)\,dt$$

$$= \int_0^3 3t^2 + \cancel{6t^3} + 6t^6 - \cancel{6t^3}\,dt$$

$$= \int_0^3 3t^2 + 6t^6\,dt = \frac{6750}{7}$$



Differential notation

$$\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$

$C$ line from $(0,0)$ to $(3,4)$ $\quad \vec{r}(t) = (3t, 4t) \quad 0 \le t \le 1$

$$\int_C y\,dx + xy\,dy = \int_0^1 4t\cdot 3\,dt + 12t^2 4\,dt$$

$$= \int_0^1 12t + 48t^2\,dt = 22$$

Ex: $\vec{F}(x,y) = x\vec{i} + y\vec{j}$ force field

Wire lying along the path $\vec{r}(t) = (2t, 3t)$ $\quad 0 \le t \le 2$.

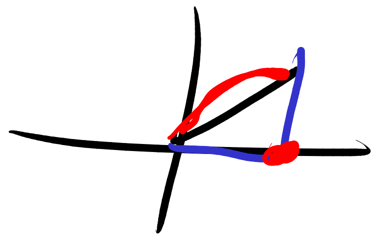
Total force on the wire?

$$\vec{F}_{tot} = \int_0^2 \vec{F}(\vec{r}(t))\, \|\vec{r}\,'(t)\|\, dt$$

$$= \int_0^2 (2t\vec{i} + 3t\vec{j})\, t\sqrt{13}\, dt = \int_0^2 2\sqrt{13}\, t^2\, dt\, \vec{i} + \int_0^2 3\sqrt{13}\, t^2\, dt\, \vec{j}$$

$$= \frac{16\sqrt{13}}{3}\vec{i} + 8\sqrt{13}\,\vec{j}.$$

# § 7.3 Conservative Vector Fields



Ex! $\vec{F}(x,y) = y\vec{i} + x\vec{j}$ $\qquad$ $C =$ line $(0,0)$ to $(1,1)$

$\vec{r}(t) = (t, t) \quad 0 \le t \le 1$

$$\int_0^1 (t,t)\cdot(1,1)\,dt = \int 2t\,dt = 1.$$

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$C_2$ $(0,0)$ to $(1,0)$ to $(1,1)$ $\qquad$ $\vec{r}_1(t) = (t, 0)$, $\vec{r}_2(t) = (1, t)$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^1 (0,t)\cdot(1,0)\,dt + \int_0^1 (t,1)\cdot(0\ 1)\,dt = 0 + \int_0^1 1\,dt = 1.$$

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$C_3$ circle centeed at $(1,0)$ $\quad$ $\vec{r}(\cos t + 1, -\sin t)$ $\quad$ $\pi \le t \le \frac{3\pi}{2}$

$$\int_{C_3} \vec{F}\cdot d\vec{r} = \int_\pi^{3\pi/2} (-\sin t, \cos t + 1)\cdot(-\sin t, -\cos t)\,dt = \int_\pi^{3\pi/2} \sin^2 t - \cos^2 t - \cos t\,dt$$

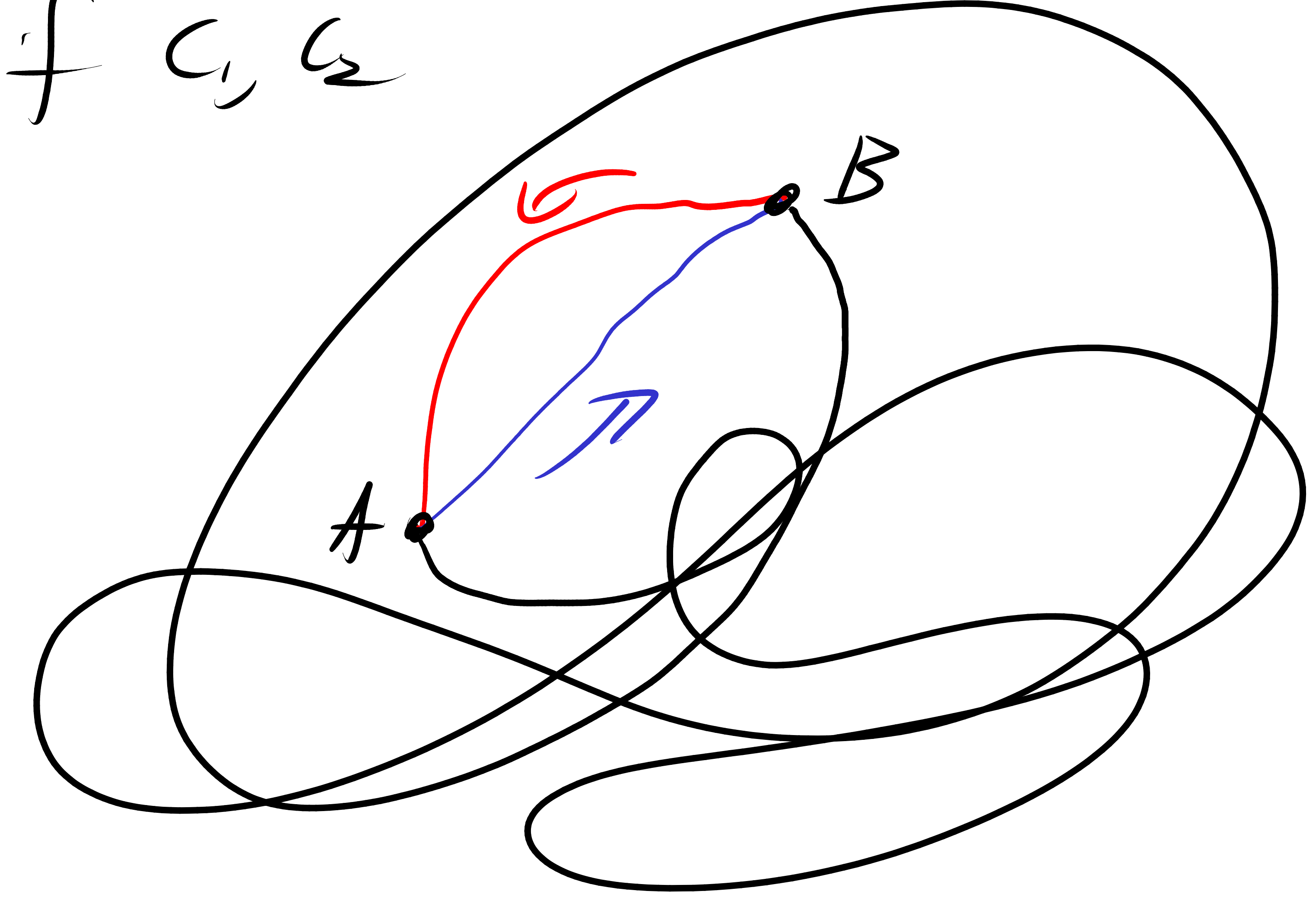
$$= 1$$

Dfn: $\vec{F}$ is <u>conservative</u> or <u>path-independent</u>

if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot dr$ if $C_1, C_2$ have same endpts.

We say $C$ is closed if start = end

$\vec{F}$ is conservative iff $\int_C \vec{F} \cdot dr = 0$ for any closed $C$.




$\vec{r}(t) = (0,0)$