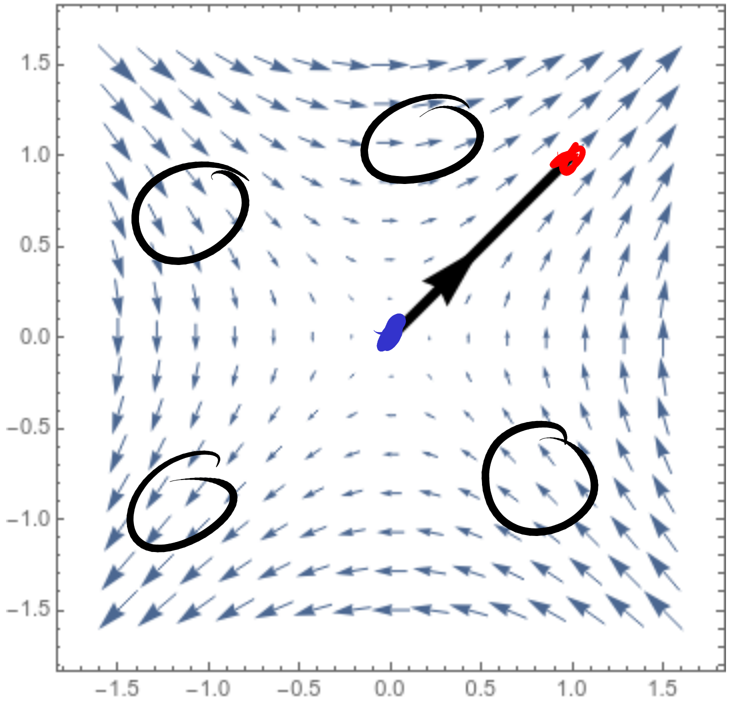
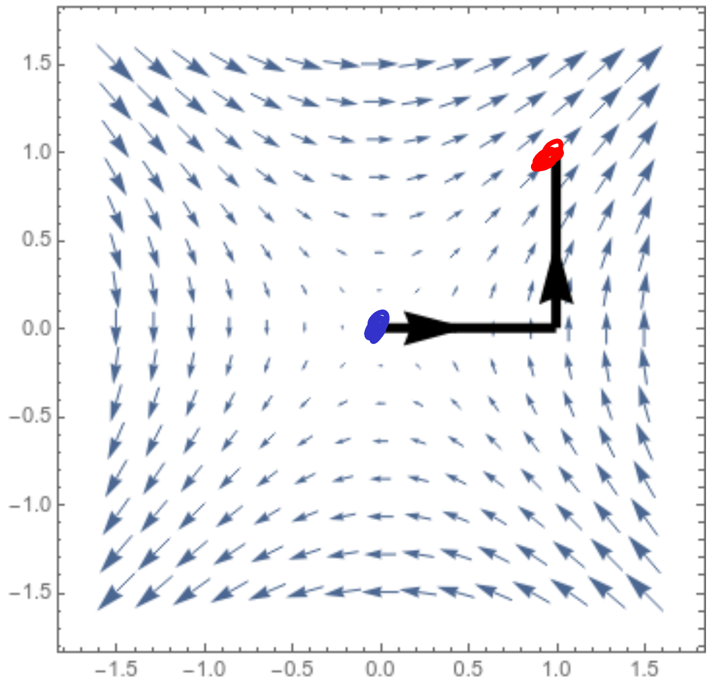
# Conservative Vector fields

VF: $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$

circulation along path

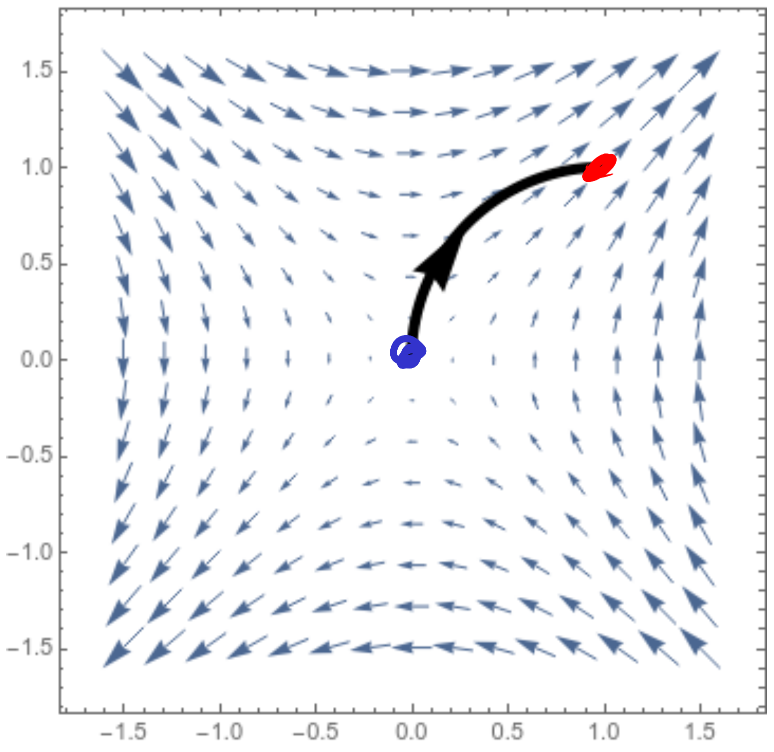
work done by the VF





Dfn: $\vec{F}$ is conservative or path-ind if $\int_C \vec{F} \cdot d\vec{r}$ depends only on start and end pts.

All the same integral



Ex: $\vec{F}(x,y) = y\vec{i} + x\vec{j}$, $\vec{r}(t) = (t + \sin^8(\pi t)\cos^5(t), \cos^4(2\pi t) + 2t)$

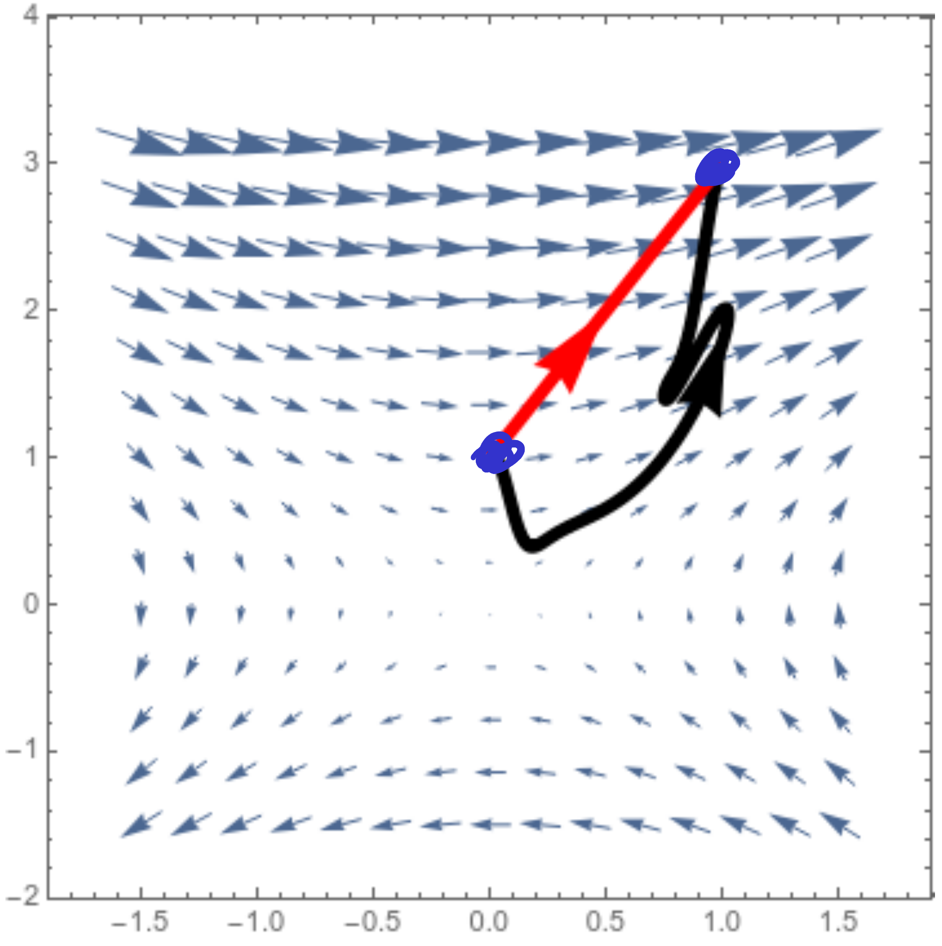
$0 \le t \le 1$

$\vec{r}(0) = (0,1)$, $\vec{r}(1) = (1,3)$

$\vec{s}(t) = (t, 1+2t)$

$\int_0^1 \vec{F}(\vec{s}(t)) \cdot \vec{s}\,'(t)\, dt$

$= \int_0^1 (1+2t, t) \cdot (1,2)\, dt = \int_0^1 1 + 4t\, dt = 3.$




$\int_C \vec{F} \cdot d\vec{r} = 3$ b/c $\vec{F}$ is conservative.

Prop (Fundamental Theorem of Calculus for line integrals):

$C$ a piecewise smooth oriented path from $P$ to $Q$

$f: \mathbb{R}^n \to \mathbb{R}$ cts diff'able on $C$. Then:

$$\int_C \nabla f \cdot d\vec{r} = f(Q) - f(P).$$

In particular $\nabla f$ is conservative.

if $\vec{F} = \nabla f$

then $f$ is a <u>potential fn</u>

for $\vec{F}$.

$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

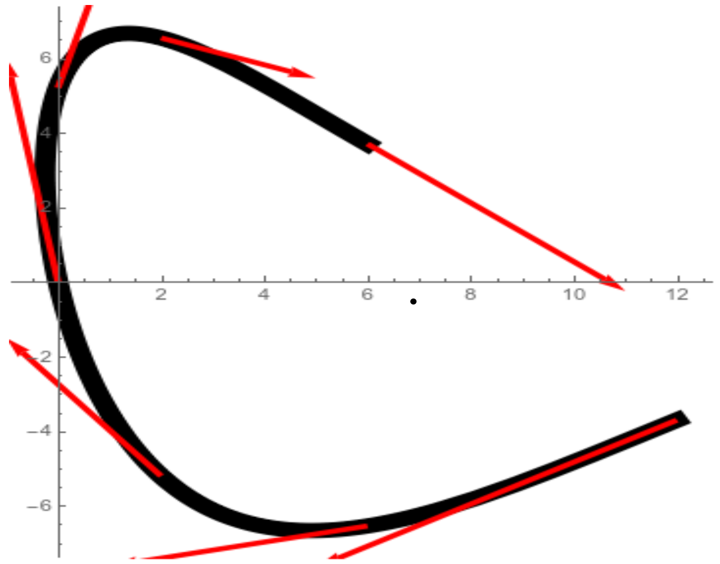
Pf/ $f(Q) - f(P)$ is $\Delta f$

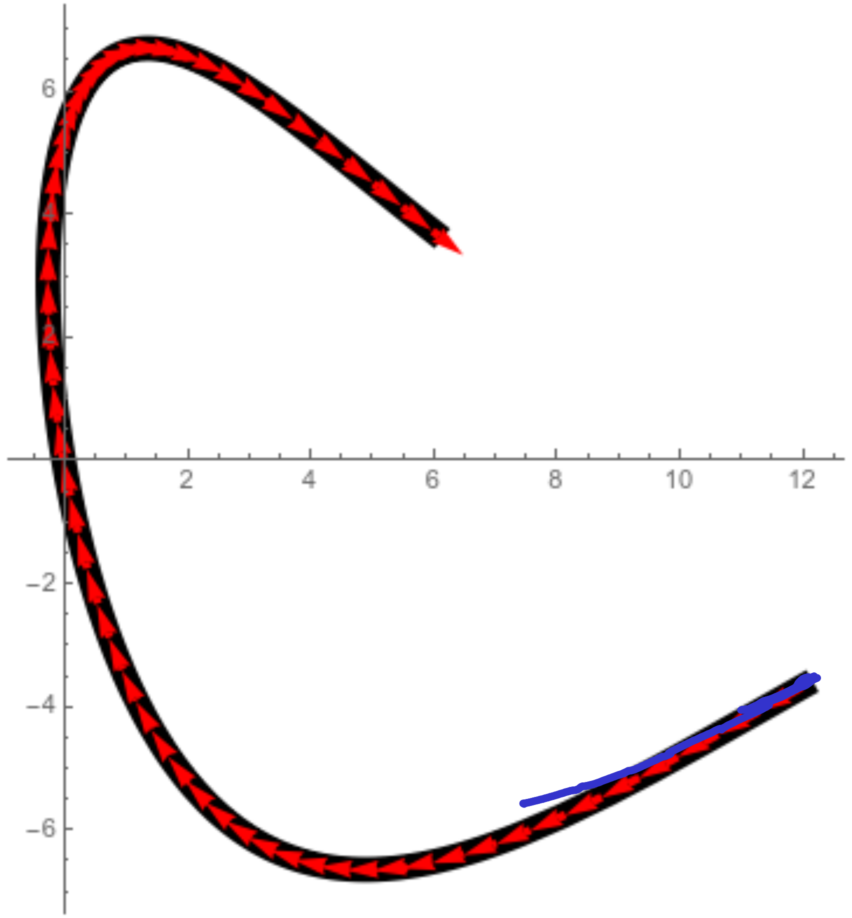
$$f(Q) \approx f(P) + \nabla f(P)\,(Q-P)$$

$$f(Q) - f(P) \approx \sum_{i=1}^{n} \nabla f(\vec{r}(t_i)) \cdot \left(\vec{r}(t_{i+1}) - \vec{r}(t_i)\right)$$

$$= \sum_{i=1}^{n} \nabla f(\vec{r}(t_i)) \cdot \frac{\vec{r}(t_{i+1} - t_i) - \vec{r}(t_i)}{\Delta t}\,\Delta t$$

$$\approx \sum_{i=1}^{n} \nabla f(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i)\,\Delta t = \int_C \nabla f \cdot d\vec{r}$$



$\vec{F}(x,y) = y\vec{i} + x\vec{j} = \nabla(xy)$

$\int_C \vec{F} \cdot d\vec{r} = f(1,1) - f(0,0) = 1 \cdot 1 - 0 \cdot 0 = 1.$

Ex: $f(x,y) = x^2 y - y^2$ $\quad$ C spiral begins at $(1,2)$
ends at $(3,1)$

$\int_C \nabla f \cdot d\vec{r} = f(3,1) - f(1,2)$

$= (9-1) - (2-4) = 8 + 2 = 10.$

FTC: $\int_a^b F'(t)\,dt = F(b) - F(a)$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Prop: $\vec{F}$ cons VF. define $f: \mathbb{R}^n \to \mathbb{R}$ by:
fix pt. $P$. Let $C$ be a path $P \to Q$. Define

$$f(Q) = \int_C \vec{F}\cdot d\vec{r}.$$

$$\vec{F} = \nabla f.$$

Cor: $\vec{F}$ cons iff
$\vec{F} = \nabla f$ for some $f$.

Ex: $\vec{F}(x,y) = y\cos x\ \vec{i} + (\sin x + y)\vec{j}$

Want $f: \mathbb{R}^2 \to \mathbb{R}$ s.t. $\frac{\partial}{\partial x} f = y\cos x$

so $f(x,y) = y\sin x + g(y)$

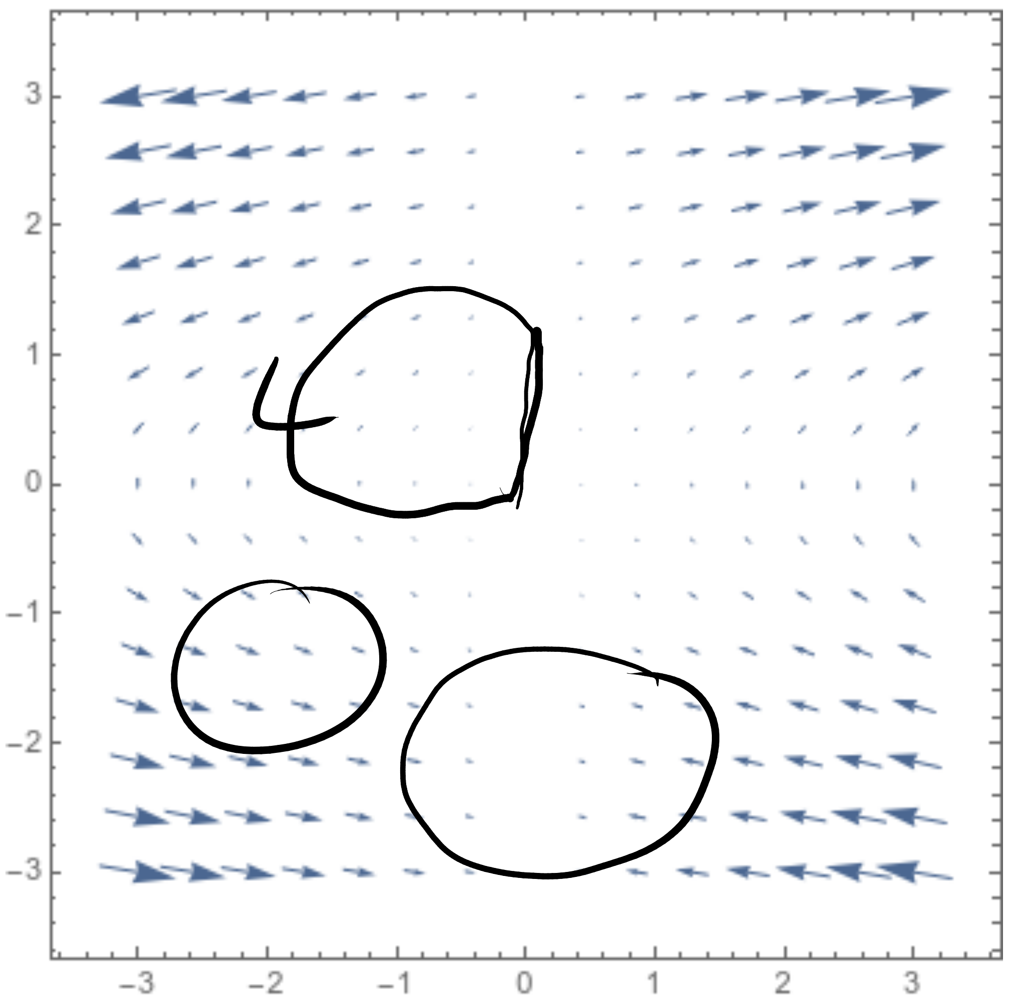
$$\frac{\partial f}{\partial y} = \sin x + y$$

$$f(x,y) = y\sin(x) + \frac{y^2}{2} + h(x)$$

Set $f(x,y) = y\sin(x) + \frac{y^2}{2} + C$

$$\cancel{\vec{F}(x,y) = \left(e^{x^2} + \sin^5(x)\cos^3(y)\,x^9\sqrt{x^4/y}\right)\vec{i} + (\;)\,\vec{j}}$$

$$\vec{F}(x,y) = 2y\,\vec{i} + x\,\vec{j}$$




Want $f(x,y)$ s.t.

$$\frac{\partial f}{\partial x} = 2y \qquad\qquad \frac{\partial f}{\partial y} = x$$

$$f(x,y) = 2xy + g(y) \qquad\qquad f(x,y) = xy + h(x)$$

$\Rightarrow\Leftarrow$ not conservative.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$ fcts diff'able

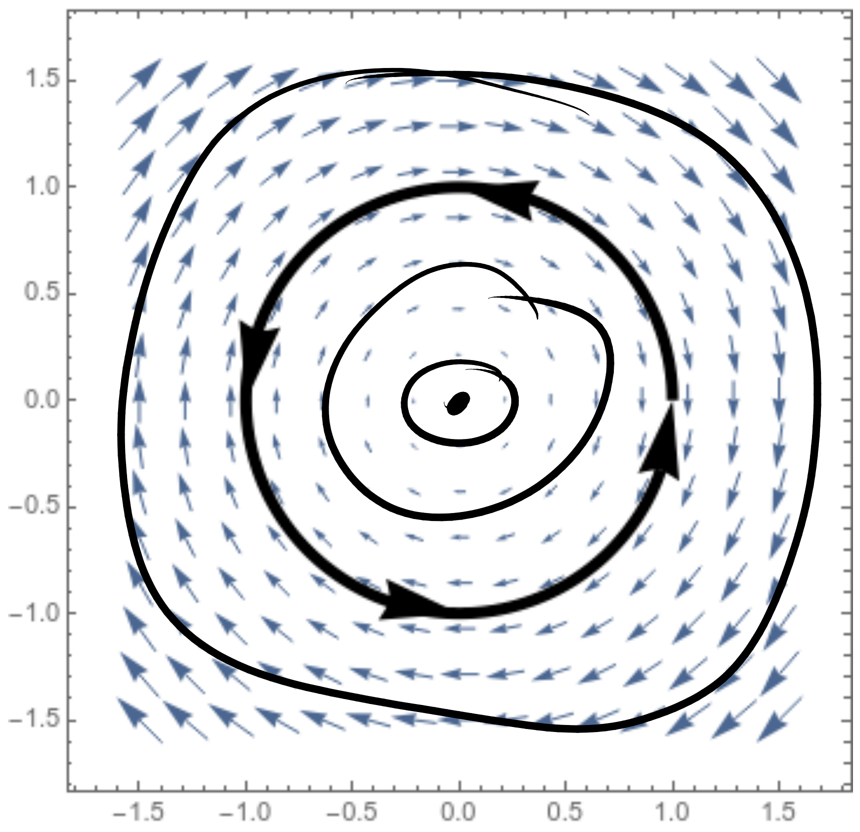
$$\frac{\partial^2 f}{\partial y \partial x} = 2 \neq \frac{\partial^2 f}{\partial x \partial y} = 1$$

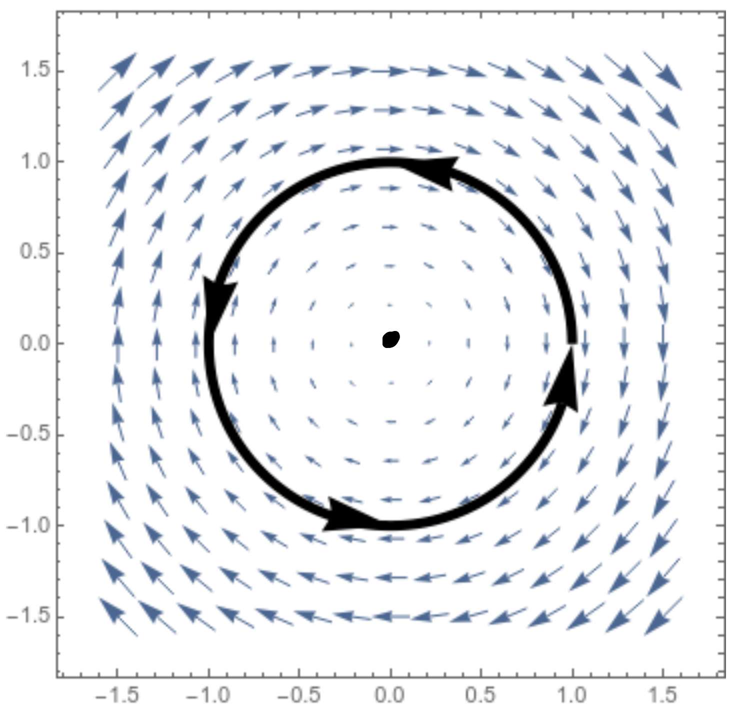
# §7.4 The curl of a VF

Dfn: The <u>circulation density</u> at $(x,y,z)$ of $\vec{F}$ around the unit vector $\vec{n}$ is

$$\text{circ}_{\vec{n}}\, \vec{F}(x,y,z) = \lim_{\text{area}\to 0} \frac{\int_C \vec{F}\cdot d\vec{r}}{\text{area of } C}$$

C a circle $\perp \vec{n}$ oriented by RHrule.






$$\vec{F}(x,y) = y\vec{i} - x\vec{j}$$

$$\vec{r}_a(t) = (a\cos t, a\sin t)$$

$$\int_0^{2\pi} (a\sin t, -a\cos t)\cdot(-a\sin t, a\cos t)\,dt$$

$$= \int_0^{2\pi} a^2(-\sin^2 t - \cos^2 t)\,dt = \int_0^{2\pi} -a^2\,dt = -2\pi a^2$$

$$\text{circ}\,(0,0,0) = \frac{\int_C \vec{F}\cdot dr}{\text{area } C} = \frac{-2\pi a^2}{\pi a^2} = -2.$$