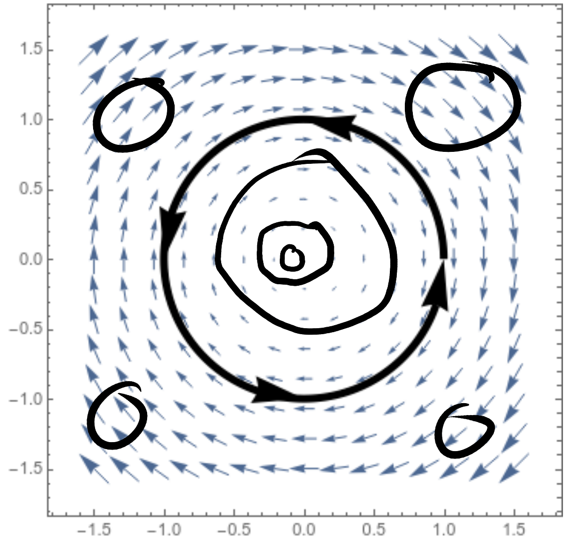
Circulation Density

$$\mathrm{circ}_{\vec{n}} \vec{F}(x,y,z) = \lim_{\substack{\text{Area}\to 0 \\ C \perp \vec{n}}} \frac{\int_C \vec{F}\cdot d\vec{r}}{\text{area } C}$$

a derivative

If $\vec{F}(x,y) = y\vec{i} - x\vec{j}$, then

$$\mathrm{circ}_{\vec{k}}(0,0) = \lim_{a\to 0} \frac{-2\pi a^2}{\pi a^2} = -2.$$



$$\frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$\nabla f$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Dfn: Let $\vec{F}(x,y,z)$

$$= F_1(x,y,z)\vec{i} + F_2(x,y,z)\vec{j} + F_3(x,y,z)\vec{k}$$

The curl of $\vec{F}$ is

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

If $\vec{F}(x,y) = F_1\vec{i} + F_2\vec{j}$

$$\nabla \times F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \quad (\vec{k})$$

Treat this as a scalar.

Direction of curl: $\vec{n}$ that maximizes $\text{circ}_{\vec{n}}\vec{F}$

$|\nabla \times \vec{F}|$ is circ density in that direction

$$\underbrace{(\nabla \times \vec{F})}_{\text{vector}} \cdot \vec{n} = \text{circ}_{\vec{n}} F$$

$$\vec{F}(x,y) = y\,\vec{i} - x\,\vec{j}$$

$$\nabla \times \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -1 - 1 = -2.$$

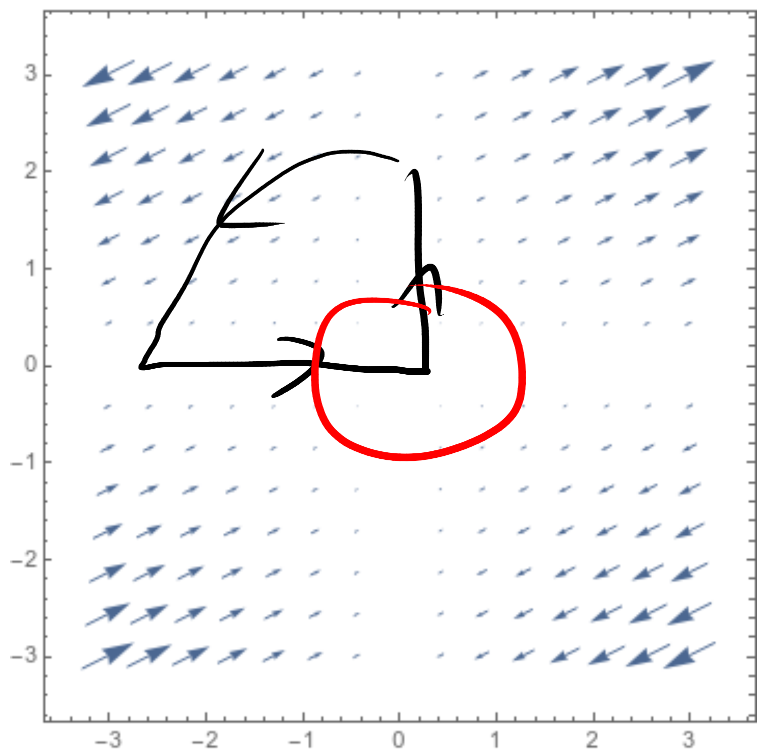
Prop: $\vec{F}$ a VF. If $\vec{F}$ is cons, then $\nabla \times \vec{F} = \vec{0}$.

If $\vec{F}$ is defined everywhere, and $\nabla \times \vec{F} = \vec{0}$, then $\vec{F}$ is cons.

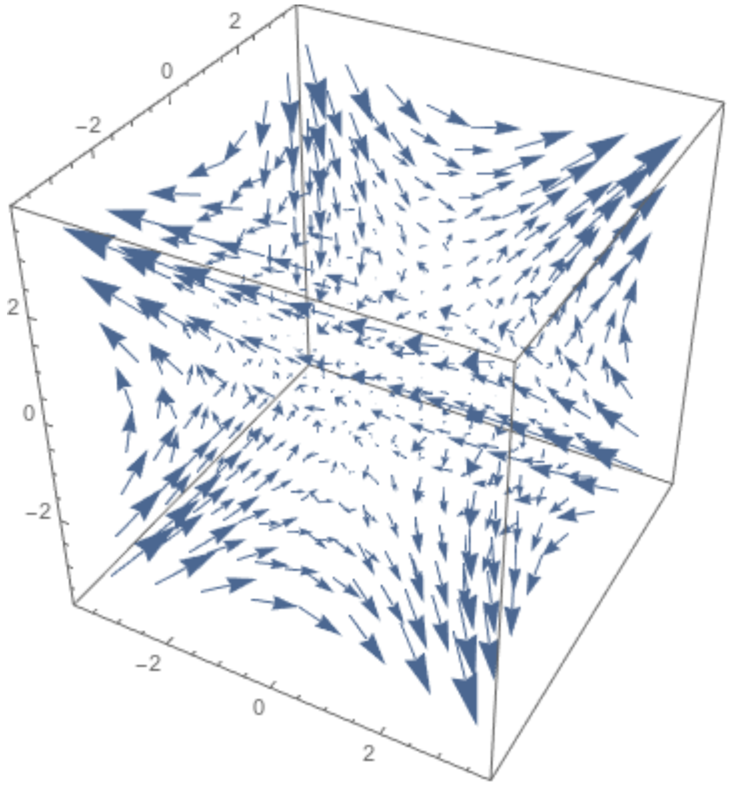
Ex: $\vec{F}(x,y) = 2xy\,\vec{i} + xy\,\vec{j}$

$$\nabla \times \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = y - 2x \neq 0$$

So $\vec{F}$ not conservative.



$$\vec{F}(x,y,z) = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$$



$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= (x - x)\vec{i} + (y - y)\vec{j} + (z - z)\vec{k} = \vec{0}.$$

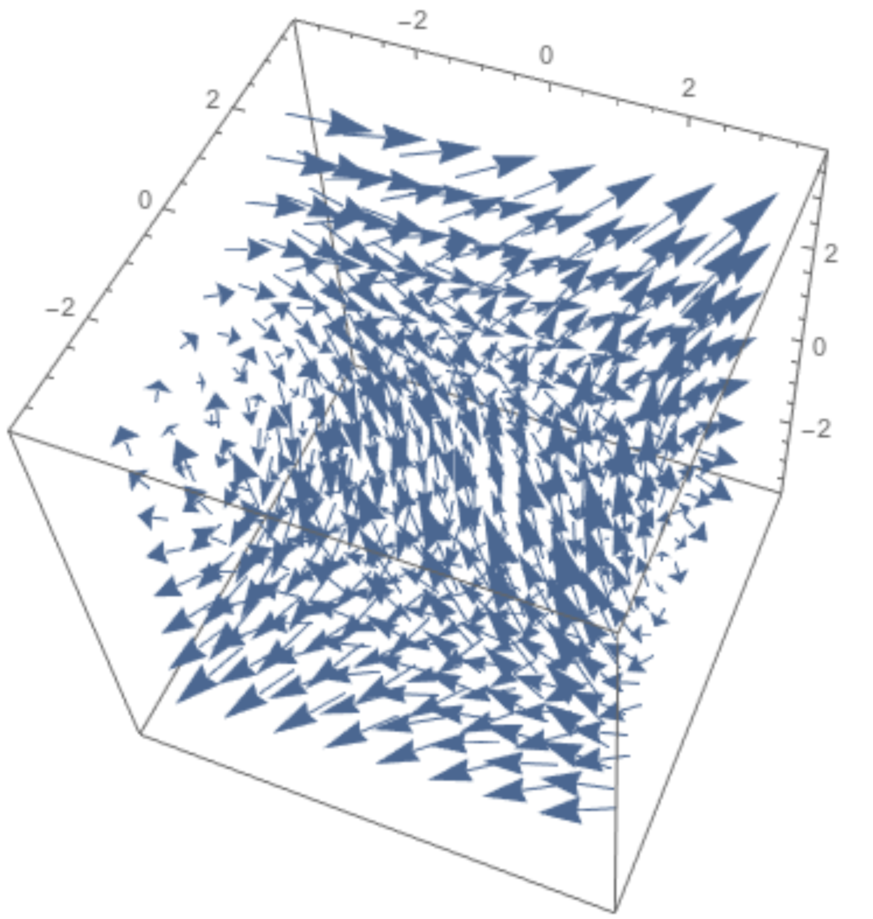
So $\vec{F}$ is conservative

$$\vec{F}(x,y,z) = y\vec{i} + z\vec{j} + x\vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

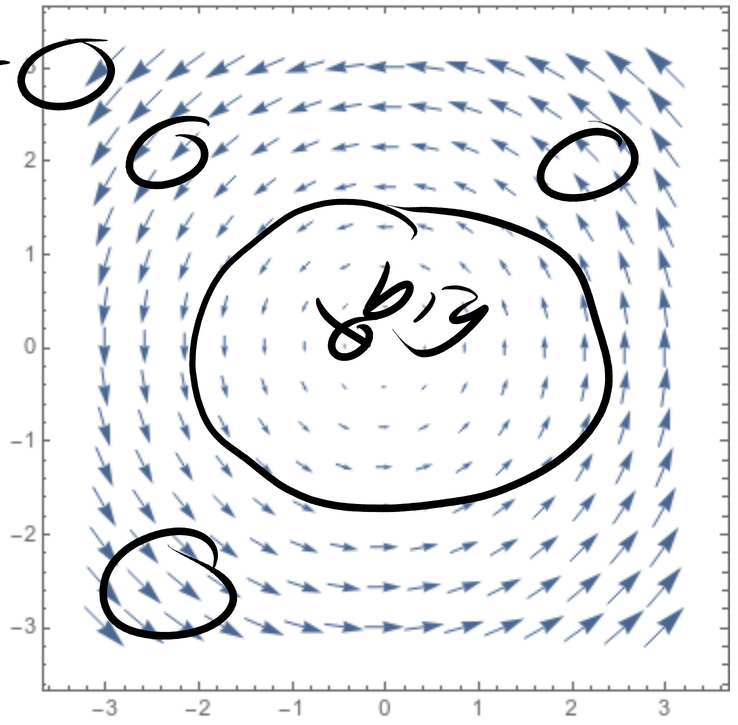
$$= (0-1)\vec{i} + (0-1)\vec{j} + (0-1)\vec{k}$$

$$= -\vec{i} - \vec{j} - \vec{k} \neq \vec{0}$$



$$\vec{F}(x,y) = \frac{-y\vec{i} + x\vec{j}}{x^2+y^2}$$

small



$-y\vec{i} + x\vec{j}$

$$\nabla \times \vec{F}(x,y) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} = 0.$$ unless $(x,y)=(0,0)$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \frac{1}{\cos^2 t+\sin^2 t}(-\sin t, \cos t)\cdot(-\sin t,\ \cos t)\, dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

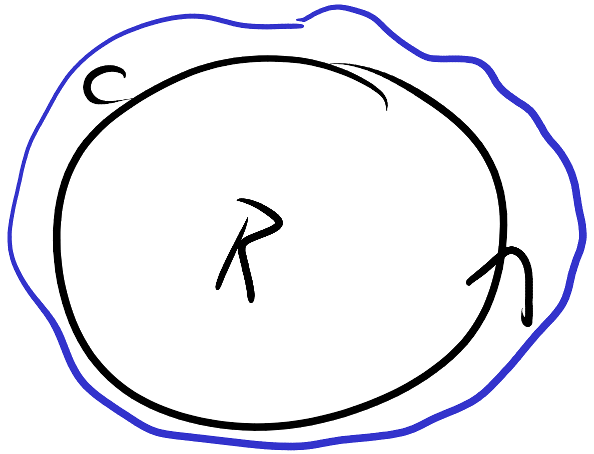
"irrotational"

circ density is 0 <u>except at origin</u>

§ 7.5 Green's Theorem

Thm (Green): C a piecewise smooth simple closed curve boundary of a region R in the plane,

$\boxed{\text{oriented so } R \text{ is to the left}}$

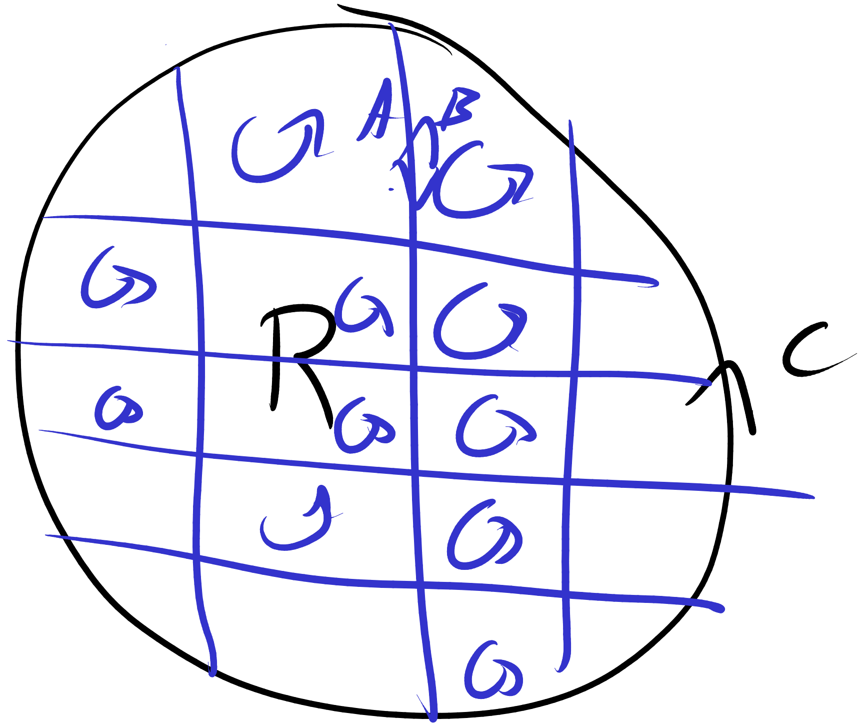


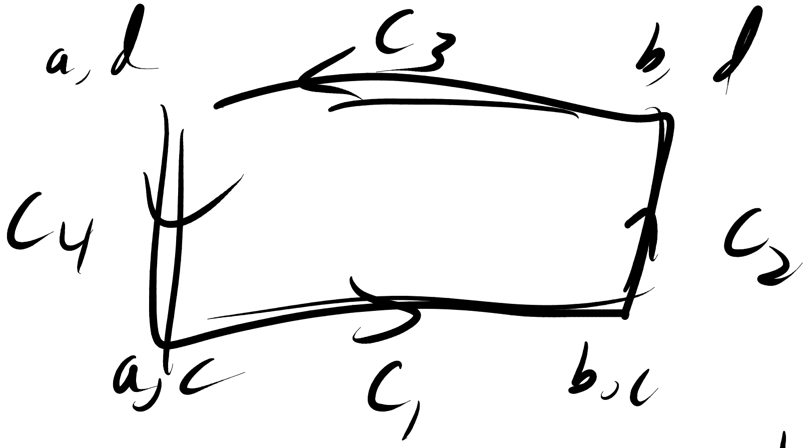
$\vec{F} = F_1 \vec{i} + F_2 \vec{j}$ a VF on an open region containing R, C. Then:

$$\int_C \vec{F} \cdot d\vec{r} = \int_R (\nabla \times \vec{F}(x,y)) \cdot \vec{k} \, dA.$$

$$\int_C F_1\, dx + F_2\, dy = \int_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dx\, dy$$

Pf/





$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_a^b \left(F_1(x,c),\ F_2(x,c)\right)\cdot(1,0)\,dx$$

$$= \int_a^b F_1(x,c)$$

$$\int_{C_3} \vec{F}\cdot dr \int_a^b -F_1(x,d)$$

$$\int_{C_1+C_3} = \int_a^b F_1(x,c) - F_1(x,d)\,dx$$

$$= \int_a^b \int_c^d -\frac{\partial F_1}{\partial y}\,dy\,dx$$

$$\int_{C_2+C_4} = \int_c^d \int_a^b \frac{\partial F_2}{\partial y}\,dy\,dx$$

Ex: $\vec{F}(x,y) = y\,\vec{i} + x^2\vec{j}$

$C =$ counterclockwise around rectangle

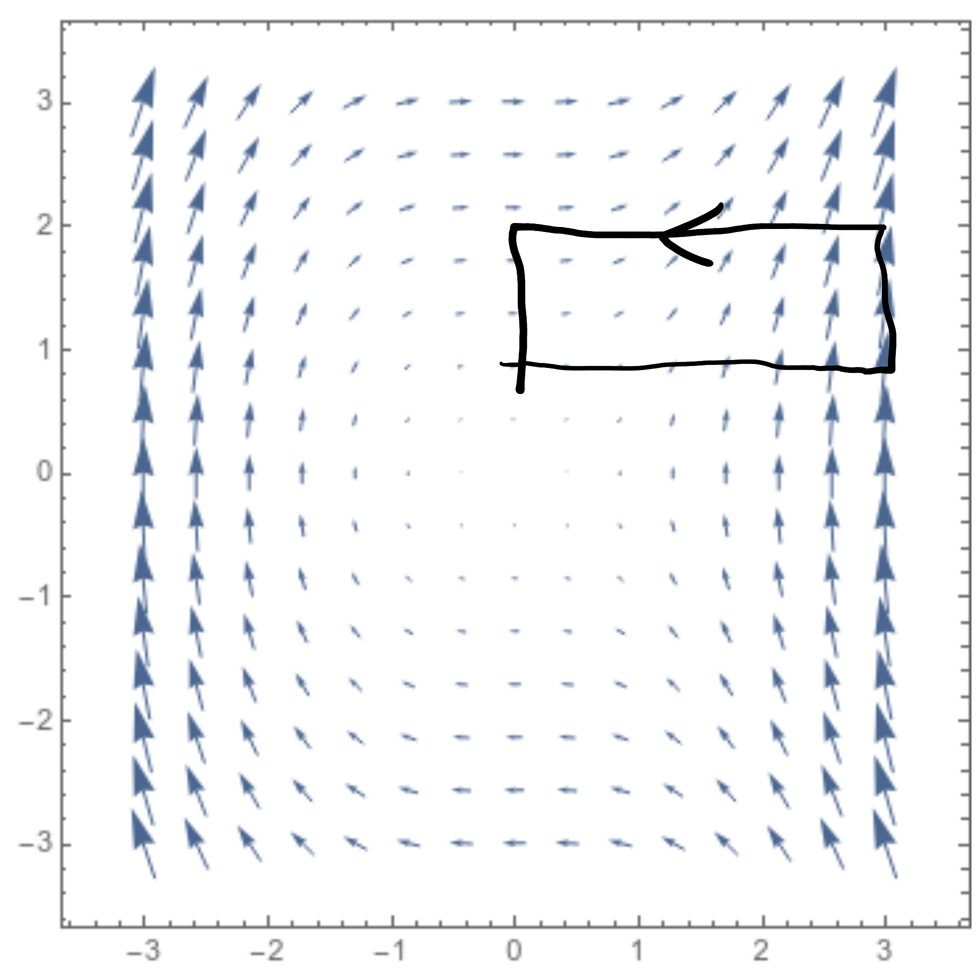
$0 \le x \le 3,\ 1 \le y \le 2$

$$\int_C \vec{F}\cdot d\vec{r} = \int_R \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\, dA$$

Gross

$$= \int_1^2 \int_0^3 2x - 1 \, dx\, dy$$

$$= \int_1^2 x^2 - x \Big|_0^3 dy = \int_1^2 6\, dy = 6.$$



$$\int_C (yx^2 - y)\,dx + (x^3+4)\,dy \qquad C \text{ goes } (0,0) \text{ to } (3,2),\ (3,0) \text{ to } (0,0)$$

$$= -\int_R \frac{\partial\, x^3+4}{\partial x} - \frac{\partial\, yx^2-y}{\partial y}\,dA$$

$$= -\int_R 3x^2 - x^2 + 1\,dA = -\int_R 2x^2+1\,dA$$

$$= -\int_0^3 \int_0^{\frac{2x}{3}} 2x^2+1\,dy\,dx = -30$$