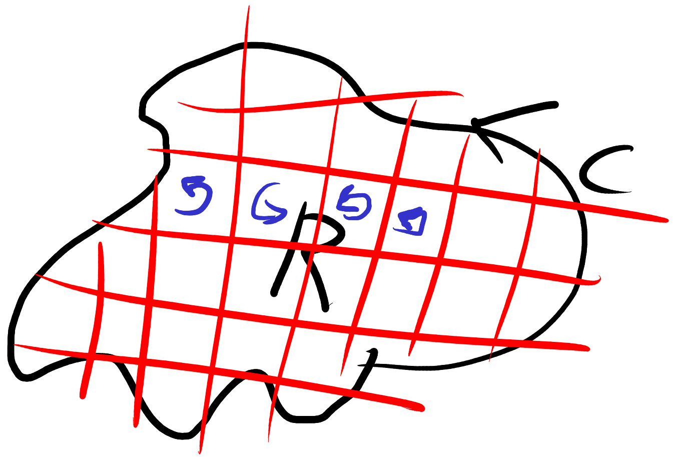
Green's Thm

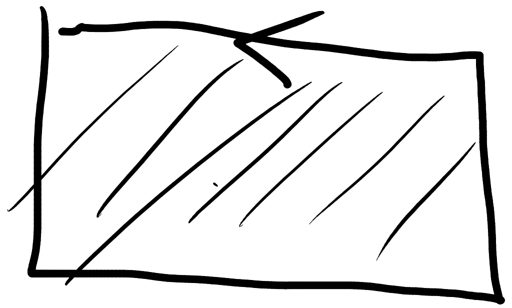
$$\int_C \vec{F}\cdot d\vec{r} = \int_R \left(\nabla\times \vec{F}(x,y)\right)\cdot \vec{k}\, dx\, dy$$

$$\int_C \vec{F}_1\, dx + \vec{F}_2\, dy = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy$$

Area of a region $\int_R 1\, dA$



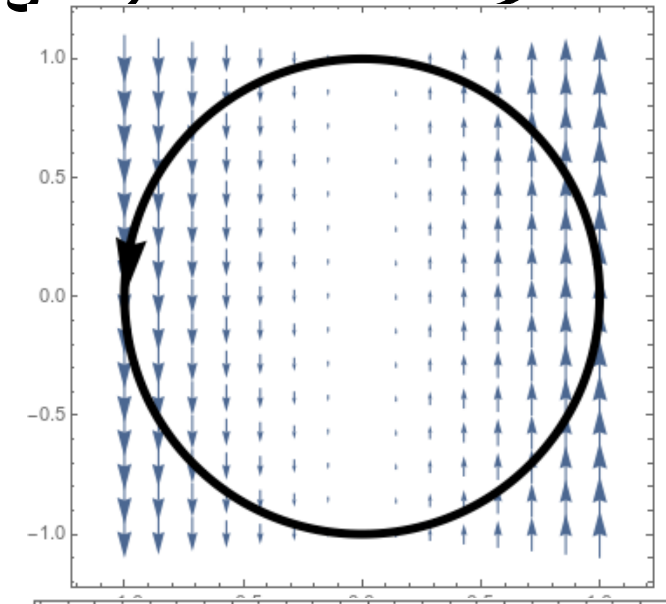
Want $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$. $\vec{F}(x,y) = x\vec{j}$, or $-y\vec{i}$, $-\frac{y}{2}\vec{i} + \frac{x}{2}\vec{j}$



$$A = \int_C x\, dy = -\int y\, dx = \frac{1}{2}\int x\, dy - y\, dx$$

Area of a region $\int_R 1 \, dA$

Find area of a circle w/ radius a. $\vec{r}(t) = (a\cos t, a\sin t)$

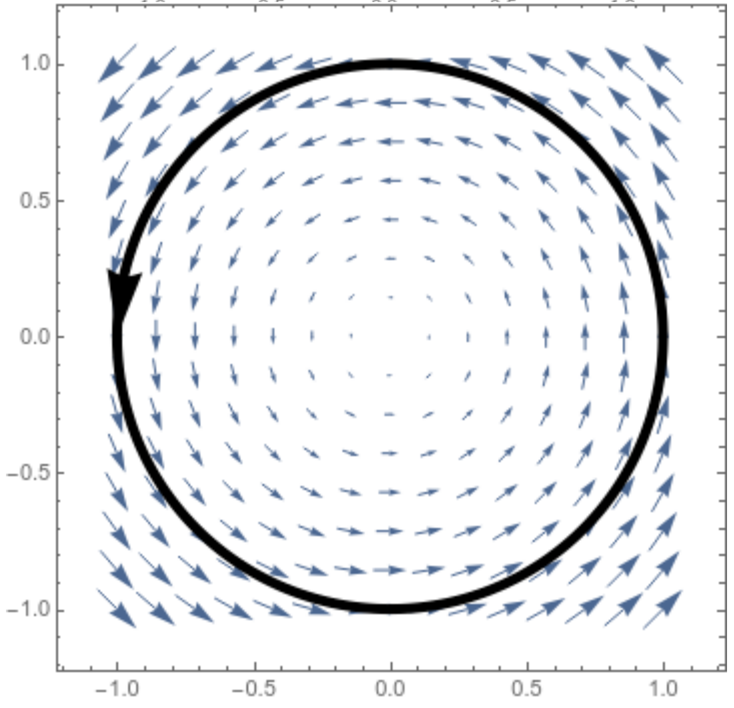


$x\vec{j}$

$A = \int_C x\,dy = \int_0^{2\pi} a\cos t \cdot a\cos t \, dt = a^2 \int_0^{2} \cos^2 t \, dt$

$\frac{1}{2} + \frac{\cos(2t)}{2}$?

$-\frac{1}{2} y\vec{i} + \frac{1}{2} x\vec{j}$



$\int_R 1 \, dA = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} a^2\cos^2 t - a\sin t(-a\sin t)\, dt$

$= \frac{a^2}{2}\int_0^{2\pi} \cos^2 t + \sin^2 t \, dt = \frac{a^2}{2}\int_0^{2\pi} 1\, dt = \pi a^2.$

§8 Surface Integrals

§8.1 Scalar Surface Integrals

Q: give density g/cm^2 of a surface, find the mass

If $\delta = 3\, g/cm^2$, surface is 3×3 square

$m = \delta \cdot A = 3 \cdot 9 = 27\ g$.

$\delta(x,y,z)$, surface is curved?

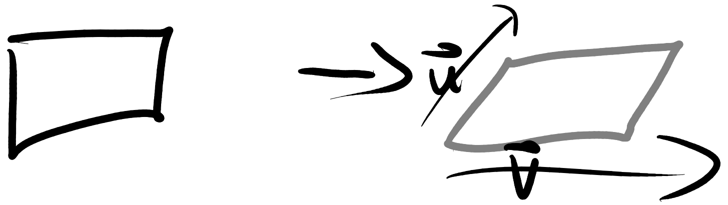
Chop surface into little rectangles

Want a parametrization

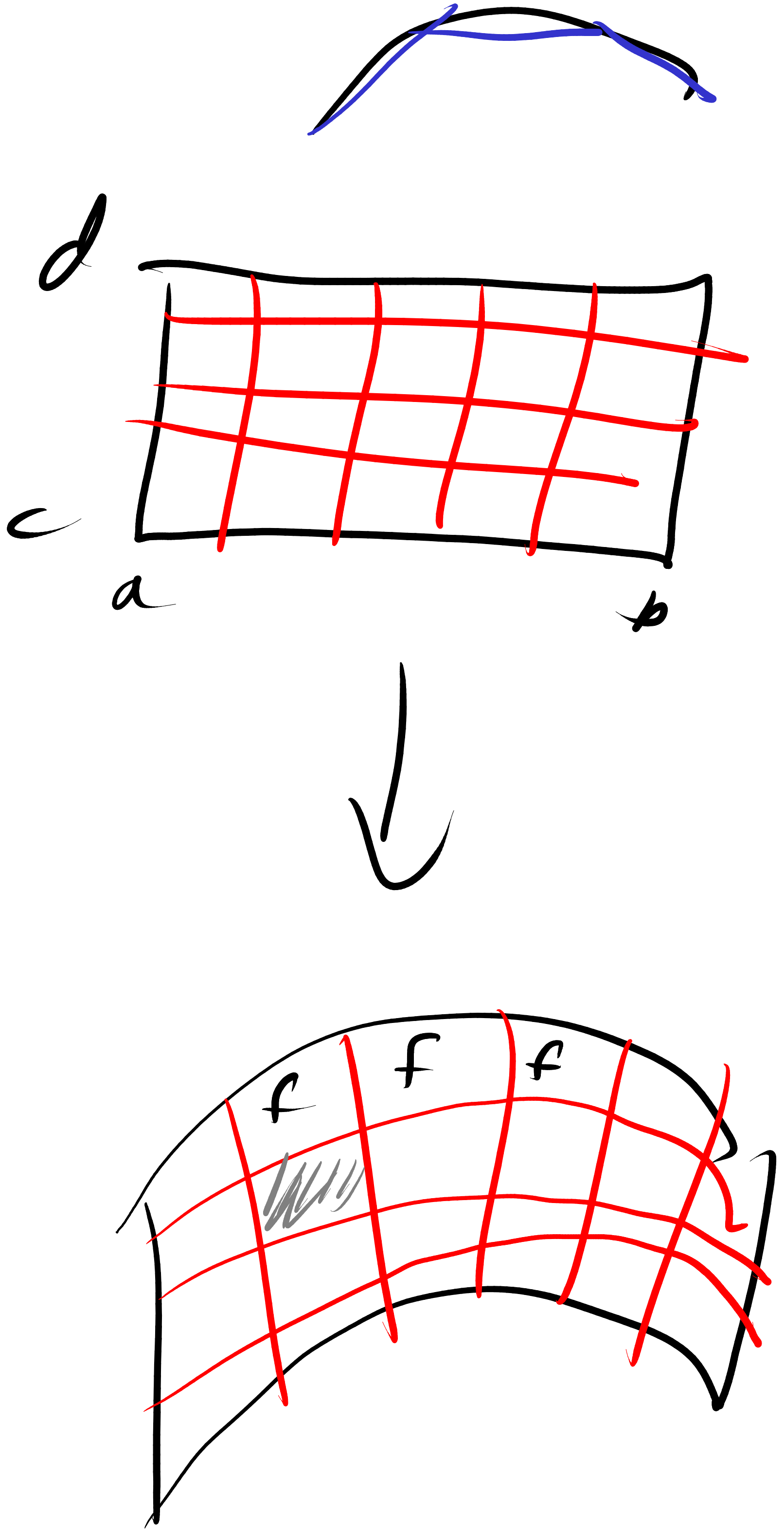
If surface is $\vec{r}(s,t)$ $\quad a \le s \le t, \quad c \le t \le d$

need value of fn on small bit: $f(x_i^*, y_i^*, z_i^*)$

need area of small bit $\|\vec{u} \times \vec{v}\|$



Area is $\|\vec{r}_s(s,t) \times \vec{r}_t(s,t)\| \Delta s \Delta t$

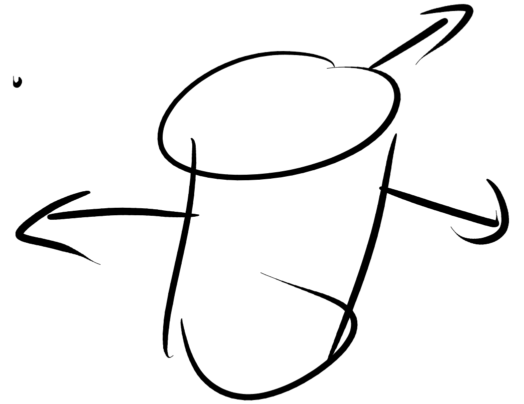


Dfn: The surface integral of $f(x,y,z)$ on S: $\vec{r}(s,t)$ over R is

$$\iint_S f\,dS = \lim_{\Delta s, \Delta t \to 0} \sum f(\vec{r}(s_i,t_i)) \, \|\vec{r}_s(s_i,t_i) \times \vec{r}_t(s_i,t_i)\| \Delta s \Delta t$$

$$= \iint_R f(\vec{r}(s,t)) \underbrace{\|\vec{r}_s(s,t) \times \vec{r}_t(s,t)\|}_{\text{Jacobian}} \, ds\, dt$$

Ex: integrate x^2 over cylinder radius 2, height 3.



$$\vec{r}(\theta, h) = (2\cos\theta,\ 2\sin\theta,\ h).$$

$$\int_R f(\vec{r}(s,t))\ \underbrace{\|\vec{r}_s(s,t) \times \vec{r}_t(s,t)\|}\ ds\,dt$$

$$\vec{r}_\theta = (-2\sin\theta, 2\cos\theta, 0)$$

$$\vec{r}_h = (0, 0, 1)$$

$$\|\vec{r}_\theta \times \vec{r}_h\| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \|2\cos\theta\,\vec{i} + 2\sin\theta\,\vec{j} + 0\vec{k}\| = \sqrt{4\cos^2\theta + 4\sin^2\theta + 0} = 2$$

$$\int_0^{2\pi}\int_0^3 (2\cos\theta)^2 \cdot 2\ dh\,d\theta = 24\int_0^{2\pi} \cos^2\theta\ d\theta = 24\pi.$$

Find mass of hemisphere $z \geq 0$ radius 1 density z g/cm²

$$\vec{r}(\theta,\phi) = (\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi) \qquad \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi/2 \end{array}$$

$$\vec{r}_\theta(\theta,\phi) = (-\sin\theta\sin\phi,\ \cos\theta\sin\phi,\ 0)$$

$$\vec{r}_\phi(\theta,\phi) = (\cos\theta\cos\phi,\ \sin\theta\cos\phi,\ -\sin\phi)$$

$$\|\vec{r}_\theta \times \vec{r}_\phi\| = \left\| -\cos\theta\sin^2\phi\,\vec{i} - \sin\theta\sin^2\phi\,\vec{j} - (\sin^2\theta\sin\phi\cos\phi + \cos^2\theta\sin\phi\cos\phi)\vec{k} \right\|$$

$$= \left\| -\cos\theta\sin^2\phi\,\vec{i} - \sin\theta\sin^2\phi\,\vec{j} - (\sin\phi\cos\phi)\vec{k} \right\|$$

$$= \sqrt{\cos^2\theta\sin^4\phi + \sin^2\theta\sin^4\phi + \sin^2\phi\cos^2\phi} = \sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi} = \sqrt{\sin^2\phi}$$

$$= |\sin\phi|$$

Find mass of hemisphere $z \geq 0$ radius 1 density z g/cm²

$\bar{r}(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$

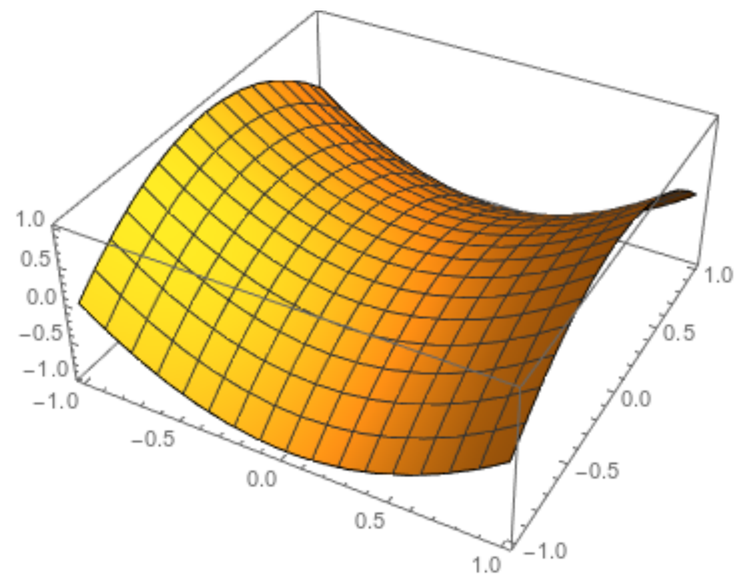
$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi/2$

$$\int_0^{2\pi} \int_0^{\pi/2} \cos\phi \, |\sin\phi| \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \cos\phi \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \frac{\sin^2\phi}{2} \Big|_0^{\pi/2} d\theta = \pi$$

Find SA of graph of $z = x^2 - y^2$ $\quad -1 \le x, y \le 1$



$\iint_S 1 \, dS \qquad \vec{r}(x,y) = (x, y, x^2 - y^2)$

$\vec{r}_x = (1, 0, 2x) \qquad \vec{r}_y = (0, 1, -2y)$

$$\|\vec{r}_x \times \vec{r}_y\| = \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} \right\| = \| -2x\vec{i} - 2y\vec{j} + \vec{k} \| = \sqrt{1 + 4x^2 + 4y^2}$$

$$\int_{-1}^{1} \int_{-1}^{1} 1 \cdot \sqrt{1 + 4x^2 + 4y^2} \, dy \, dx = 4 - \frac{1}{3}\arctan(4/3) + \frac{7}{3}\ln(5) \approx 7.45$$

Prop: sphere radius r $\vec{r}(\theta, \phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$

Then $\|\vec{r}_\theta \times \vec{r}_\phi\| = r^2|\sin\phi|$

Cylinder of radius r $\vec{r}(\theta, h) = (r\cos\theta, r\sin\theta, h)$

Then $\|\vec{r}_\theta \times \vec{r}_h\| = r$

Graph of $z = f(x,y)$ $\vec{r}(x,y) = (x, y, f(x,y))$

$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{\frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2 + 1}$