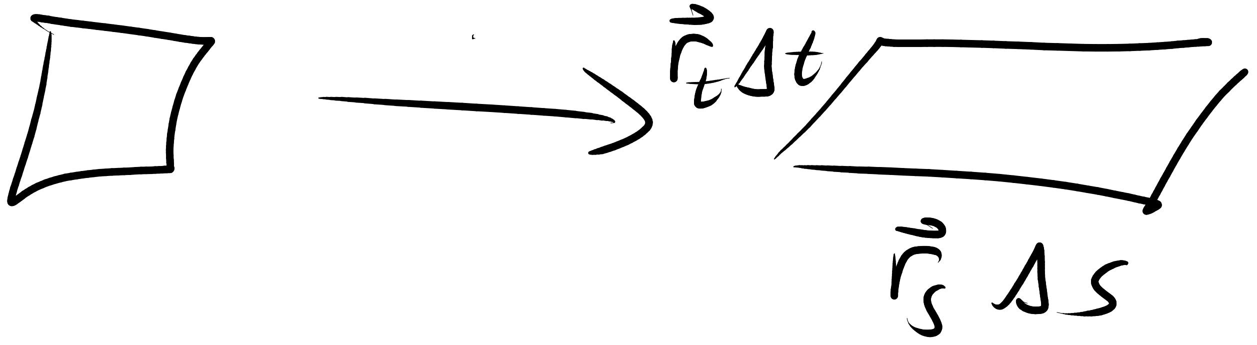
Scalar surface integrals

$$\int_S f\, dS = \int_a^b \int_c^d f(\vec{r}(s,t))\, \|\vec{r}_s(s,t) \times \vec{r}_t(s,t)\|\, ds\, dt$$

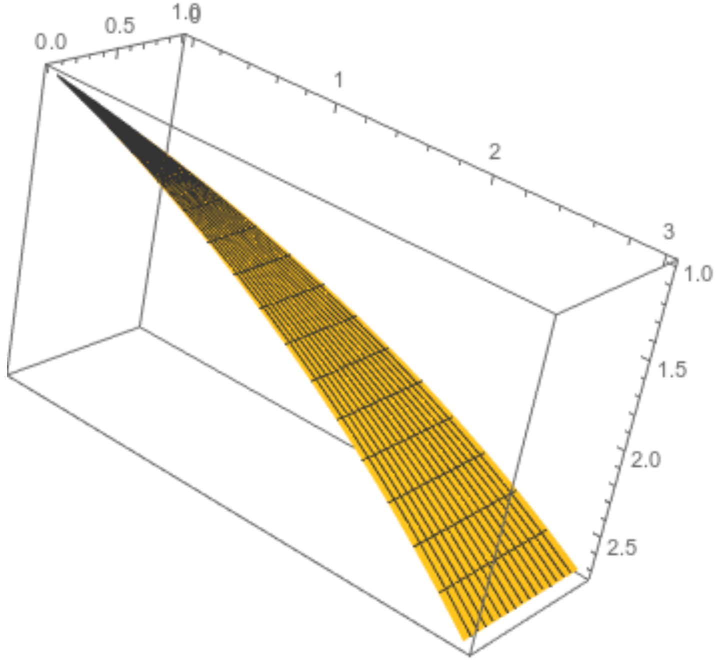


Ex integrate $\frac{x^2}{y}$ over surface $\vec{r}(s,t) = (e^s, st, 3s)$

$1 \le s, t \le 2$

$1 \le s \le 5$

$3 \le t \le 4$



$\vec{r}_s(s,t) = (e^s, t, 3)$

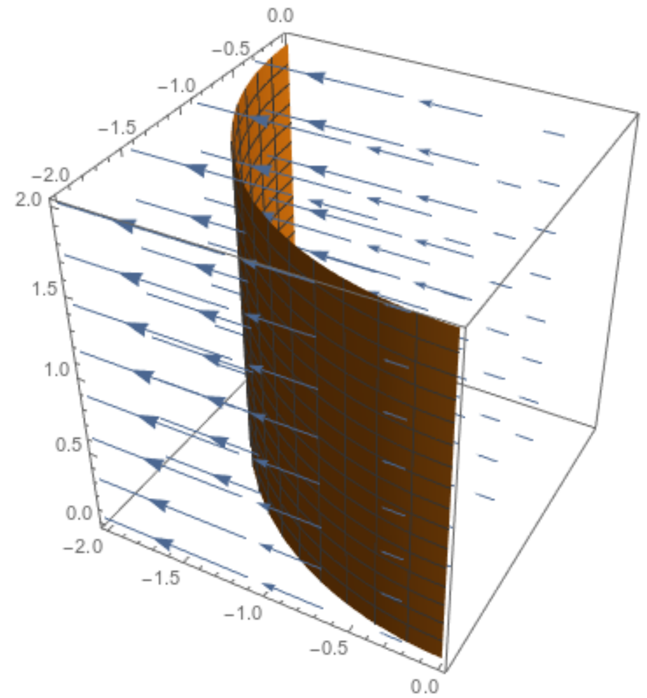
$\vec{r}_t(s,t) = (0, s, 0)$

$$\vec{r}_s \times \vec{r}_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ e^s & t & 3 \\ 0 & s & 0 \end{vmatrix} = -3s\vec{i} + se^s\vec{k}$$

$$\int_{1 \atop 3}^{2 \atop 4} \int_1^{2 \atop 5} \frac{e^{2s}}{\cancel{s}t} \sqrt{9s^{\cancel{2}} + \cancel{s}^2 e^{2s}} \, ds \, dt = \cdots =$$

$$\left((9+e^4)^{3/2} - (9+e^2)^{3/2} \right) \frac{\ln(2)}{3}$$

§8.2 Flux integrals

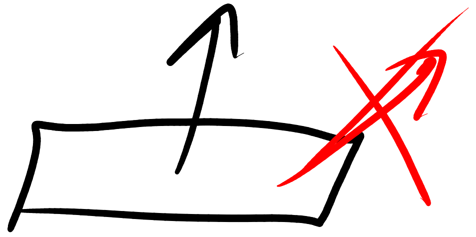


Dfn: The *orientation* of a surface is a cts choice of normal vector at each pt.

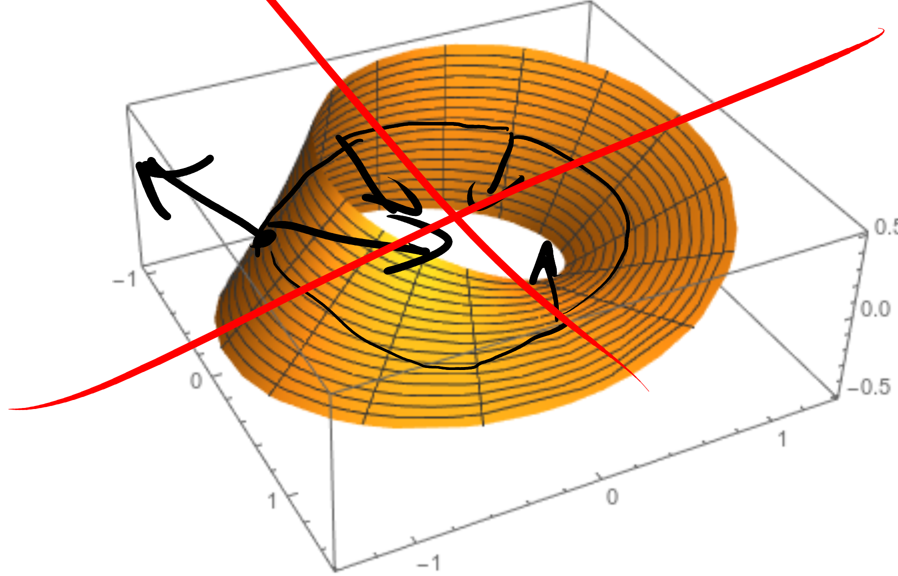
The *area vector* is a vector $\vec{A}$

direction of orientation

magnitude: SA of the surface.



not orientable



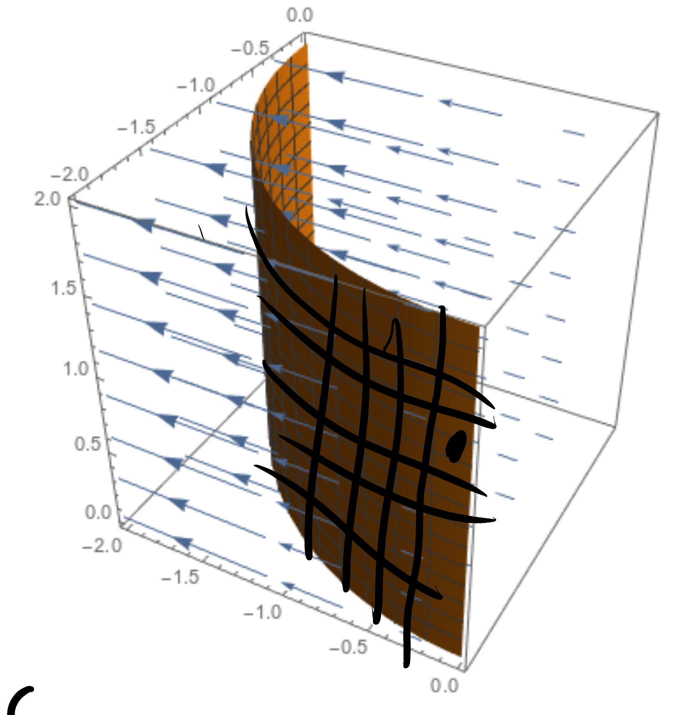
the flux of a vector $\vec{v}$ through S

is $\vec{v} \cdot \vec{A}$.

Easy w/ constant $\vec{v}$, flat $\vec{S}$.

What if $\vec{v} = \vec{F}(x,y,z)$, S weird

Break into pieces ss constant ss flat



Dfn the <u>Flux Integral</u> of $\vec{F}$ through $\vec{S}$ is

$$\int_S \vec{F} \cdot d\vec{A} = \lim_{\|\Delta \vec{A}\| \to 0} \sum \vec{F} \cdot \Delta \vec{A}$$

If $\vec{S}$ is a closed surface oriented outwards

call this the flux out of $\vec{S}$.

How to compute?

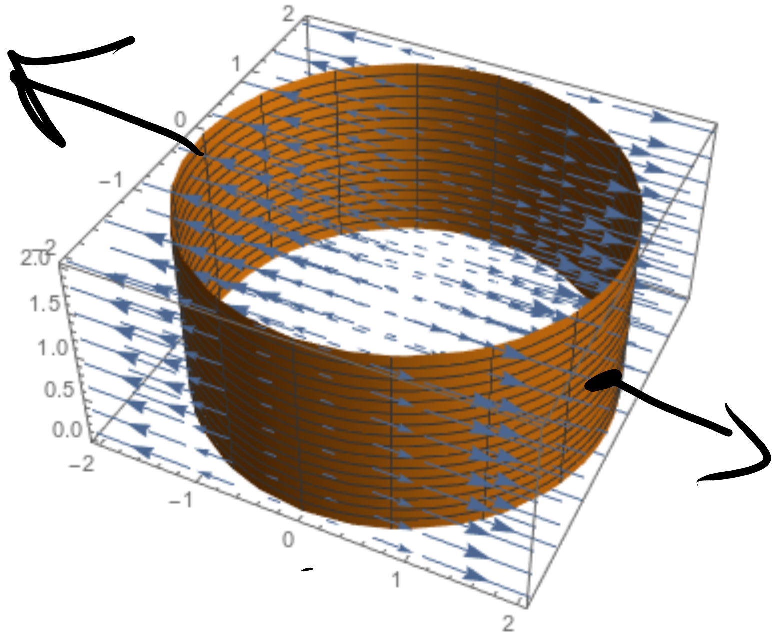
$$\sum \vec{F} \cdot \Delta \vec{A} \approx \sum \vec{F} \cdot (\vec{r}_s \times \vec{r}_t)\, \Delta s\, \Delta t$$

$$\approx \int_a^b \int_c^d \vec{F} \cdot (\vec{r}_s \times \vec{r}_t)\, dt\, ds$$

Example: $\vec{F}(x,y,z) = x\vec{i}$

flux outwards through cylinder

radius 2, $0 \le z \le 2$



$$\vec{r}(\theta, z) = (2\cos\theta, 2\sin\theta, z)$$

$$\vec{r}_\theta \times \vec{r}_z = (2\cos\theta, 2\sin\theta, 0) \qquad \vec{r}_z \times \vec{r}_\theta$$

$$\int_0^2 \int_0^{2\pi} (2\cos\theta, 0, 0) \cdot (2\cos\theta, 2\sin\theta, 0)\, d\theta\, dz$$

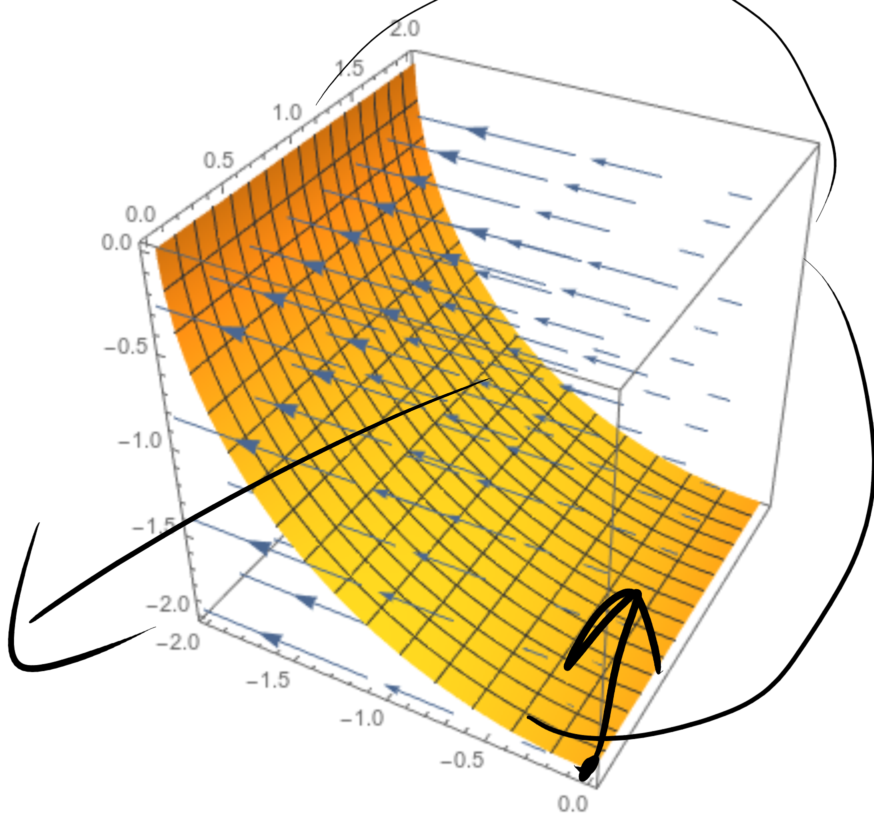
$$= \int_0^2 \int_0^{2\pi} 4\cos^2\theta \, d\theta\, dz = \int_0^2 2\theta + \sin(2\theta) \Big|_0^{2\pi} dz$$

$$= \int_0^2 4\pi \, dz = 8\pi.$$

$\vec{F}(x,y,z) = x\vec{i}$

cylinder radius 2 centered on y axis

$0 \le y \le 2, \quad x \le 0, \quad z \le 0$



$\vec{r}(\theta, y) = (2\cos\theta, y, 2\sin\theta)$

$0 \le y \le 2$

$\pi \le \theta \le 3\pi/2$

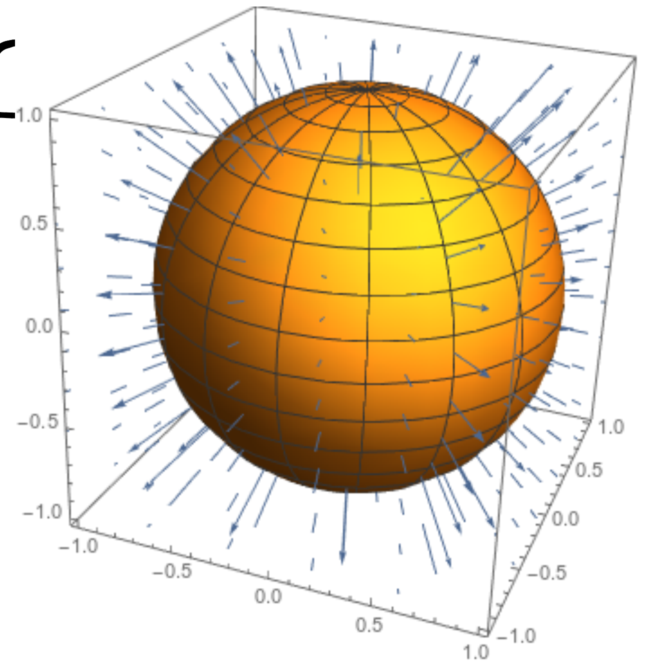
$$\vec{r}_\theta \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta & 0 & 2\cos\theta \\ 0 & 1 & 0 \end{vmatrix} = (-2\cos\theta, 0, -2\sin\theta)$$

$\vec{r}_y \times \vec{r}_\theta = (2\cos\theta, 0, 2\sin\theta)$

$$\int_0^2 \int_\pi^{3\pi/2} (2\cos\theta, 0, 0) \cdot (+2\cos\theta, 0, +2\sin\theta)\, d\theta\, dy$$

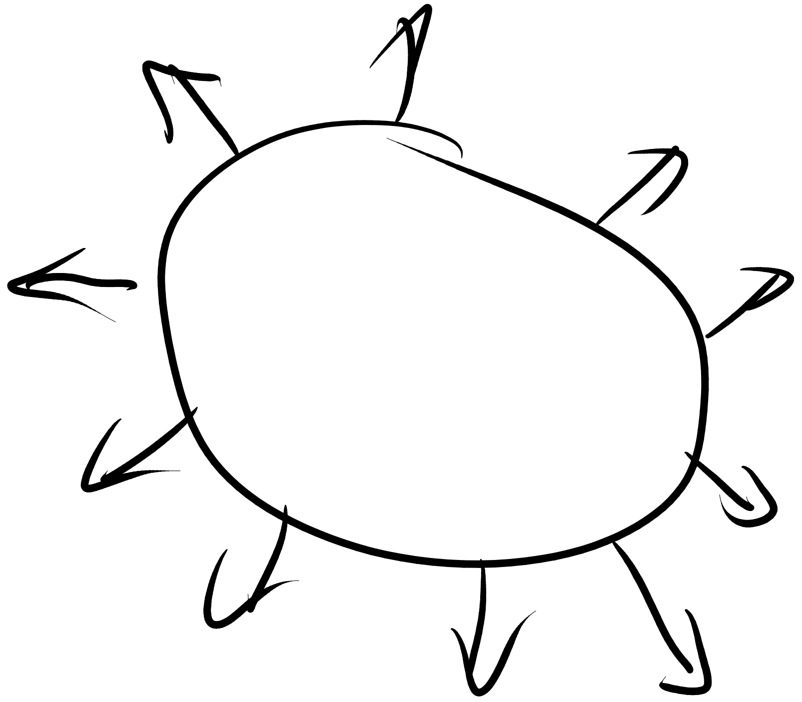
$$= \int_0^2 \int_\pi^{3\pi/2} +4\cos^2\theta\, d\theta\, dy = +2\pi$$

Ex (Gauss's Law) Flux of $\vec{F}(\vec{r}) = \dfrac{\vec{r}}{\|\vec{r}\|^3}$ out of sphere of radius R.



$$F(x,y,z) = \frac{(x,y,z)}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

$\vec{F}$ and $\vec{S}$ always $\perp$



So $\vec{F}\cdot\Delta\vec{A} = \|\vec{F}\|\,\|\Delta\vec{A}\| = \dfrac{1}{\|\vec{r}\|^2}\|\Delta A\| = \dfrac{1}{R^2}\|\Delta A\|$

$$\sum \frac{1}{R^2}\|\Delta A\| = \frac{1}{R^2}\sum\|\Delta A\| = \frac{1}{R^2}\, SA = \frac{1}{R^2}(4\pi R^2)$$

$$= 4\pi.$$

$$\vec{r}(\theta, \phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$$

$$\vec{r}_\phi \times \vec{r}_\theta = R^2\cos\theta\sin^2\phi\,\vec{i} + R^2\sin\theta\sin^2\phi\,\vec{j} + R^2\sin\phi\cos\phi\,\vec{k}$$

$$\int_0^{2\pi}\int_0^{\pi} \frac{1}{R^3}(R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)\cdot(R^2\cos\theta\sin^2\phi\vec{i} + R^2\sin\theta\sin^2\phi\vec{j} + R^2\sin\phi\cos\phi\vec{k})\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \cos^2\theta\sin^3\phi + \sin^2\theta\sin^3\phi + \sin\phi\cos^2\phi\,d\phi\,d\theta = \int\int \sin^2\phi\sin\phi + \cos^2\phi\sin\phi\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\phi\,d\phi\,d\theta = \int_0^{2\pi} -\cos\phi|0^\pi\,d\theta = \int_0^{2\pi} 2\,d\theta = 4\pi.$$

Prop: if S is graph of $z = f(x,y)$, then

$$d\vec{A} = (-f_x \vec{i} - f_y \vec{j} + \vec{k})\, dx\, dy$$

If S is cyl oriented out, then

$$d\vec{A} = R(\cos\theta \vec{i} + \sin\theta \vec{j})\, dz\, d\theta$$

If S is sphere oriented out, then

$$d\vec{A} = \frac{\vec{r}}{\|\vec{r}\|}\, dA = (\sin\phi\cos\theta \vec{i} + \sin\phi\sin\theta \vec{j} + \cos\phi\, k) R^2 \sin\phi\, d\phi\, d\theta$$

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