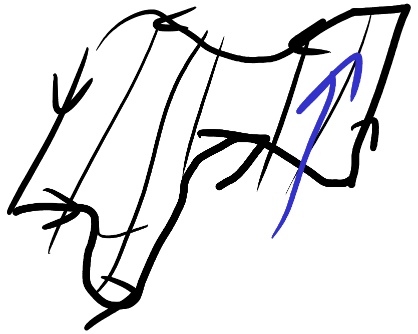
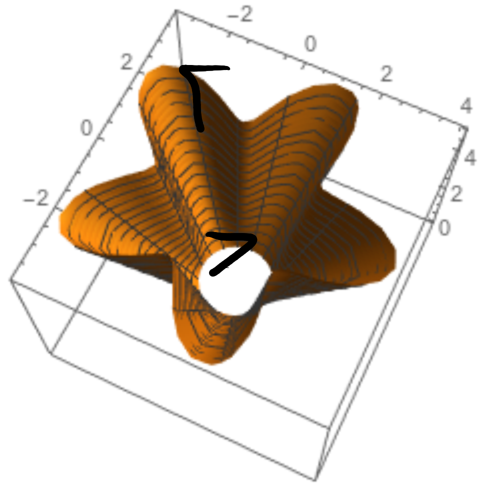
Stokes's Thm:

$$\int_C \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot d\vec{A}$$



1) Replace gross surface integral w/ line integral.

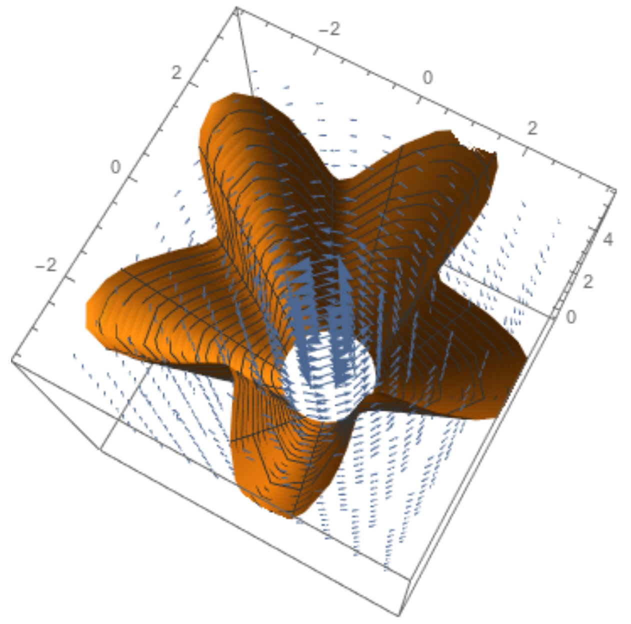
2) Replace gross line integral w/ surface integral ??



If curl is zero:

3) Replace line integral w/ easier line integral

4) Replace surface integral w/ another surface same bdry



If S, T have the same bdry C, then

$$\int_S \nabla \times \vec{F} \cdot d\vec{A} = \int_T \nabla \times \vec{F} \cdot d\vec{A}$$

I don't need to know $\vec{F}$ for this to work

just need to know that I <u>have</u> a curl field.

Dfn: G VF. <u>If</u> $\exists \vec{F}$ s.t. $\nabla \times \vec{F} = G$,

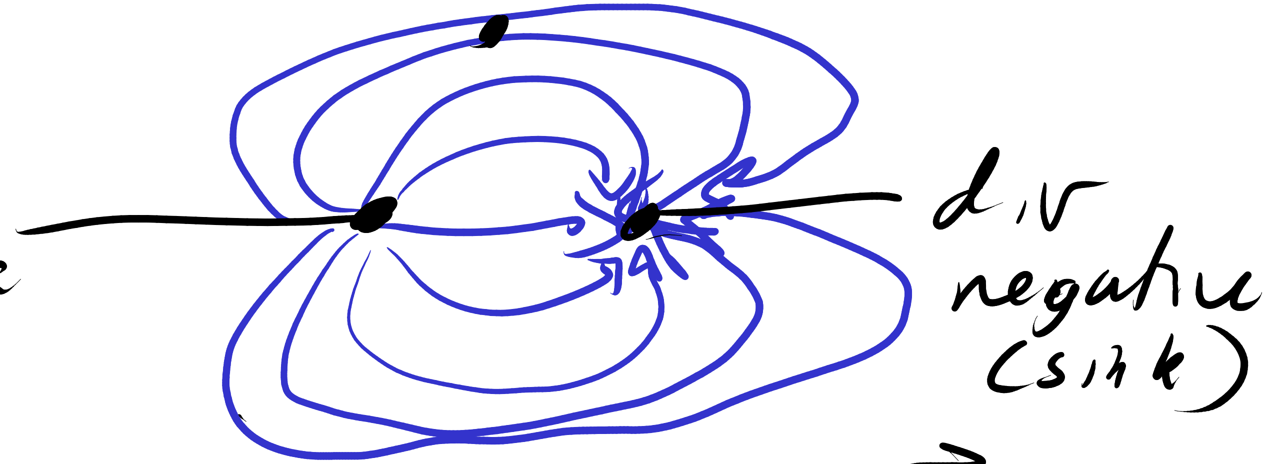
say G is a <u>curl field</u> and $\vec{F}$ is a <u>vector potential</u> for G.

Prop: If G a curl field, then 2 surfaces w/ same bdry

have the same flux integral.

§9 Divergence

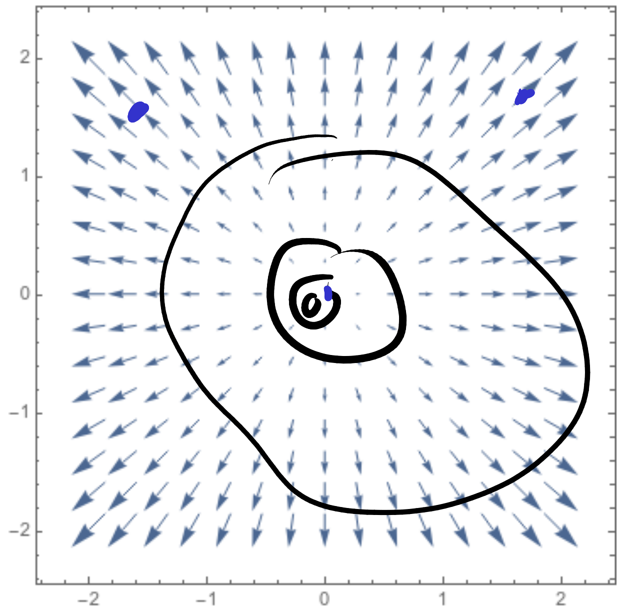
§9.1 Div of a vector field

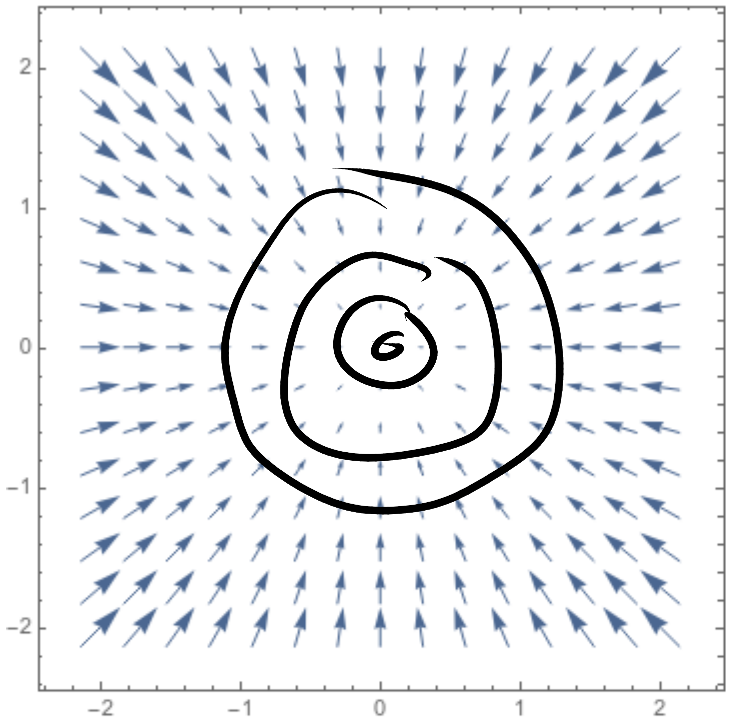


Dfn: The **divergence** or **flux density** of a VF $\vec{F}$ is

$$\nabla \cdot \vec{F}(x,y,z) = \lim_{Vol \to 0} \frac{\int_S \vec{F} \cdot d\vec{A}}{\text{Volume of } S}$$

where S is sphere centered at (x,y,z) oriented outwards.



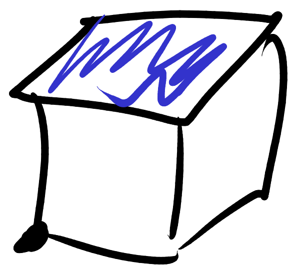


Prop: $\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}\right)$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

— amount leaving through top
— amount entering through bottom.

Δx by Δy in R



x_0, y_0, z_0

Pf/ take S to be a small box Δx by Δy by Δz

Approx $\vec{F}(x,y,z) = \vec{F}(x_0, y_0, z_0)$ on whole box.

Flux through the top $= \vec{F}(x_0, y_0, z_0 + \Delta z) \cdot (\Delta x \Delta y\, \vec{k}) = F_3(x_0, y_0, z_0+\Delta z)(z_0 + \Delta z)(\Delta x \Delta y)$

bottom $\quad \vec{F}(x_0, y_0, z_0) \cdot (\Delta x \Delta y\, (-\vec{k})) = F_3(x_0, y_0, z_0)(-z_0 \Delta x \Delta y)$

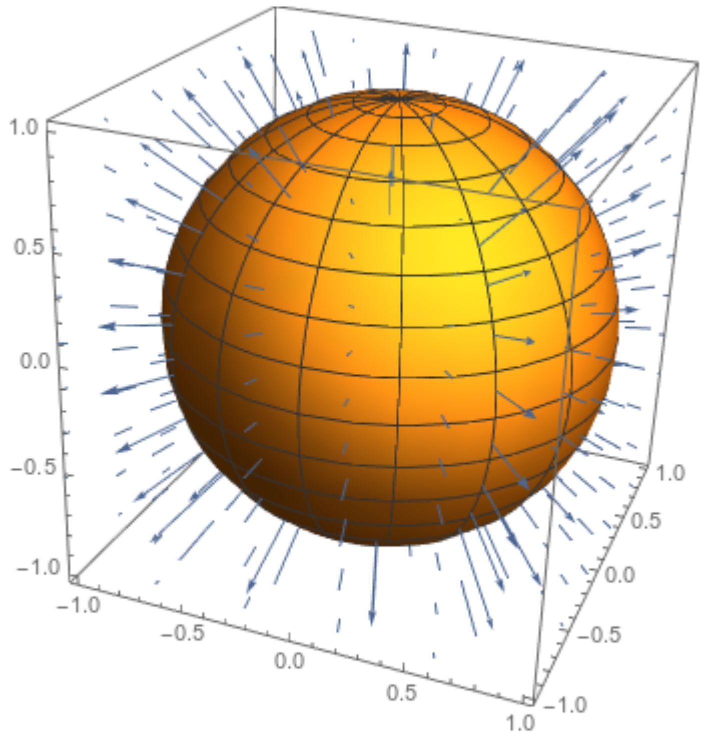
top + bottom: $\Delta x \Delta y \Delta z \underbrace{\frac{F_3(x_0, y_0, z_0+\Delta z) - F_3(x_0, y_0, z_0)}{\Delta z}}_{\partial F_3/\partial z}$

$$\vec{F}(\vec{r}) = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

find $\nabla \cdot \vec{F}$ at origin.

$$\lim_{a \to 0} \frac{\int_S \vec{F} \cdot d\vec{A}}{4/3\,\pi a^3} = \lim_{a\to 0} \frac{4\pi a^3}{\frac{4}{3}\pi a^3}$$

$$= 3$$



$$\nabla \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1 = 3.$$

$$\int_S \vec{F} \cdot d\vec{A} = \int_S \vec{F} \cdot \frac{\vec{r}}{\|\vec{r}\|}\, dA$$

$$= \int_S \vec{r} \cdot \vec{r} \, \frac{1}{\|\vec{r}\|}\, dA$$

$$= \int_S \frac{\|\vec{r}\|^2}{\|\vec{r}\|}\, dA = \int_S \|\vec{r}\|\, dA$$

$$= \|\vec{r}\| \cdot \text{SA of sphere}$$

$$= \|\vec{r}\| \cdot 4\pi \|\vec{r}\|^2 = 4\pi a^3$$

$\vec{F}(\vec{x}, \vec{y}, \vec{z}) = x^2 y \vec{i} + \cos(z)\vec{j} + \sin(z)\vec{k}$

$\nabla \cdot \vec{F} = \frac{\partial x^2 y}{\partial x} + \frac{\partial \cos(z)}{\partial y} + \frac{\partial \sin(z)}{\partial z}$

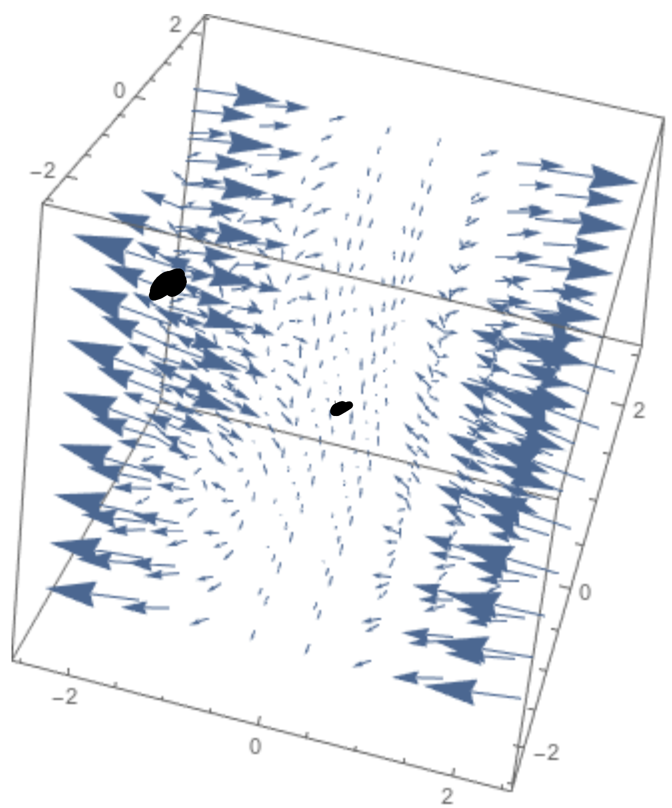
$= 2xy + 0 + \cos z$

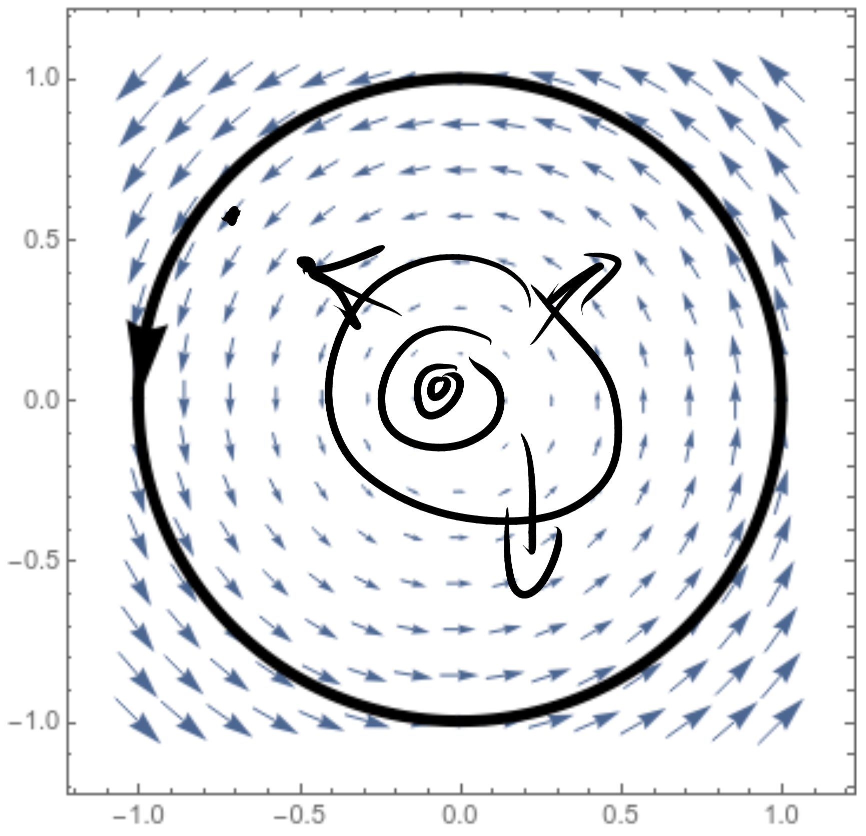
$\nabla \cdot \vec{F}(0,0,0) = 0 + 0 + \cos(0) = 1$

$G = (x,y) = -y\vec{i} + x\vec{j}$

$\nabla \cdot G = \frac{\partial -y}{\partial x} + \frac{\partial x}{\partial y} = 0 + 0 = 0.$

Div is 0 at every point





Dfn: $\vec{F}$ is divergence-free or solenoidal or incompressible if $\nabla \cdot \vec{F} = 0$ when $\vec{F}$ is defined.

"Ex" $\vec{E} = \dfrac{\vec{r}}{\|\vec{r}\|^p}$ $\quad E_1 = \dfrac{x}{(x^2+y^2+z^2)^{p/2}}$

$$\frac{\partial E_1}{\partial x} = \frac{x^2+y^2+z^2-px^2}{(x^2+y^2+z^2)^{p/2+1}}$$

$$\nabla \cdot \vec{E} = \frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z} = \frac{3-p}{\|\vec{r}\|^p}.$$

$\therefore$ Solenoidal when $p=3$.

inverse square law.

Prop: If $\vec{G} = \nabla \times \vec{F}$ is a curl field,
then $\nabla \cdot \vec{G} = 0$.

If $\nabla \cdot \vec{G} = 0$ everywhere (G is defined everywhere),
then $G = \nabla \times \vec{F}$ for some F.

Pf/ Suppose $\nabla \times \vec{F} = \vec{G}$. then

$$\nabla \cdot \vec{G} = \lim_{vol \to 0} \frac{\int_S \vec{G} \cdot d\vec{A}}{Vol} = \lim_{vol \to 0} \frac{1}{vol} \int_C \vec{F} \cdot dA$$ by stokes's.

But S has no bdry, so $\int_C \vec{F} \, dA = 0$.

Conversely, if $\nabla \cdot \vec{G} = 0$, define $F(\vec{r}) = \int_0^1 \vec{G}(t\vec{r}) \times t\vec{r} \, dt$