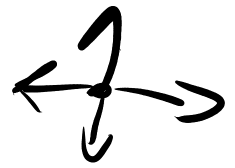
Divergence



$$\nabla\cdot\vec{F}(x,y,z)=\lim_{Vol\to 0}\frac{\int_S \vec{F}\cdot d\vec{A}}{\text{volume}}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

If $\vec{G}=\nabla\times\vec{F}$, then $\nabla\cdot\vec{G}=0$.

$(\nabla\cdot\nabla\times\vec{G}=0)$

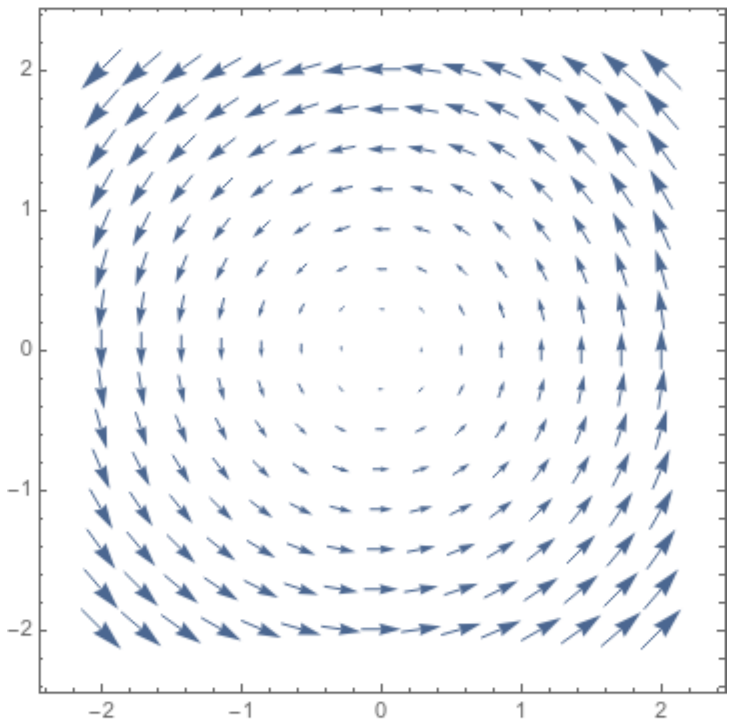
If $\nabla\cdot\vec{G}=0$ everywhere, then $\exists\,\vec{F}$ s.t. $G=\nabla\times\vec{F}$.

$$\nabla\times\nabla f=\vec{0}$$

and if $\nabla\times\vec{F}=\vec{0}$ everywhere then $\exists\, f$ s.t. $\vec{F}=\nabla f$.

$\vec{G}(x,y) = -y\vec{i} + x\vec{j}$ is divergence-free

So $\exists\ \vec{F}$ s.t. $\vec{G} = \nabla \times \vec{F}$



$$\nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = -y$$

This is hard!

If $\vec{F}$ works then $\vec{F} + \nabla f$ works for any f.

$$\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = x$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

$$\cancel{\frac{\partial F_3}{\partial y}} - \frac{\partial F_2}{\partial z} = -y$$

$$\frac{\partial F_1}{\partial z} - \cancel{\frac{\partial F_3}{\partial x}} = x$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

$$-\frac{\partial F_2}{\partial z} = -y$$

$$\frac{\partial F_1}{\partial z} = x$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

This is hard!

If $\vec{F}$ works then $\vec{F} + \nabla f$ works for any f.

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assume $\frac{\partial f}{\partial z} = -F_3$, so assume $F_3 = 0$.

$\frac{\partial F_2}{\partial z} = y$, so $F_2 = yz + g(x,y)$

$\frac{\partial F_1}{\partial z} = x$, so $F_1 = xz + h(x,y)$

$\Rightarrow \frac{\partial g}{\partial x} - \frac{\partial h}{\partial y} = 0$. Simplest: $g = h = 0$.

$\vec{F} = xz\vec{i} + yz\vec{j}$ + 0 $\vec{k}$.

$$\int f\,dx = F + C$$

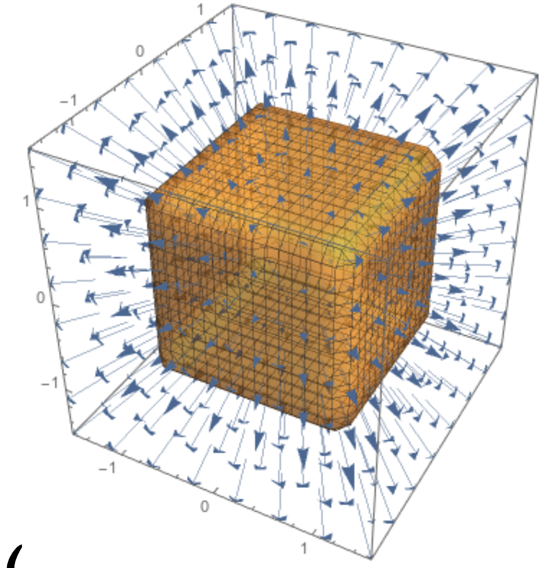
can assume $F(0) = 0$

$$\int \sin x\,dx = -\cos x + C$$

$F(0) = -1$?

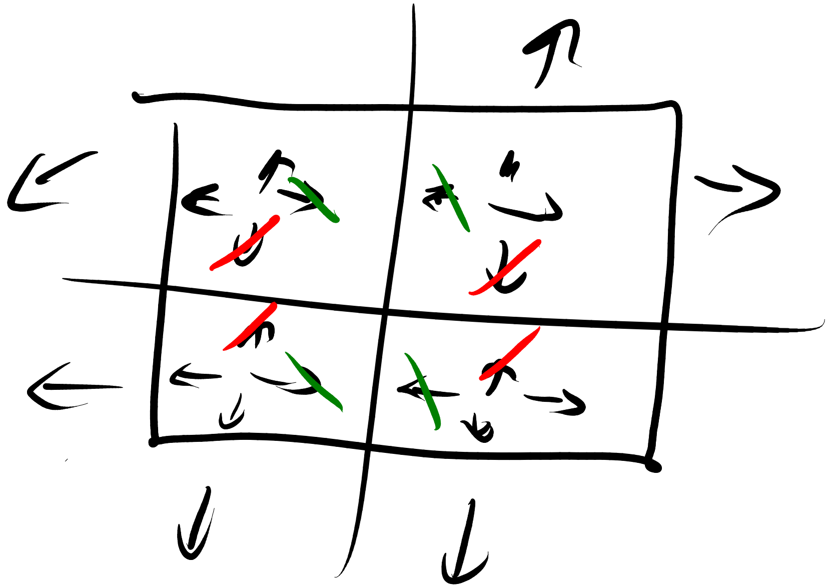
but if I take $C = 1$, then $F(0) = 0$

§9.2 Divergence Thm



Thm: Let W be a solid 3D region boundary S piecewise smooth oriented outwards. $\vec{F}$ smooth VF

$$\underbrace{\int_S \vec{F}\cdot d\vec{A}}_{\text{total Flux}} = \int_W \underbrace{\nabla\cdot\vec{F}}_{\text{flux density}}\, dV$$

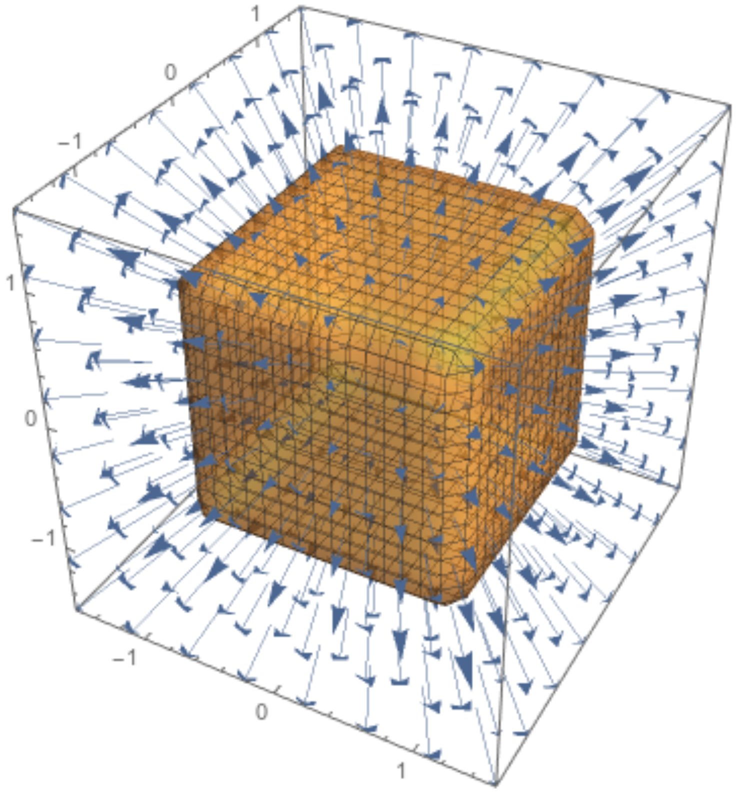


Ex: W cube side length 2

S bdry $= \partial W$

$\vec{F}(\vec{r}) = \vec{r}$, i.e. $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$

find flux of $\vec{F}$ out of S.

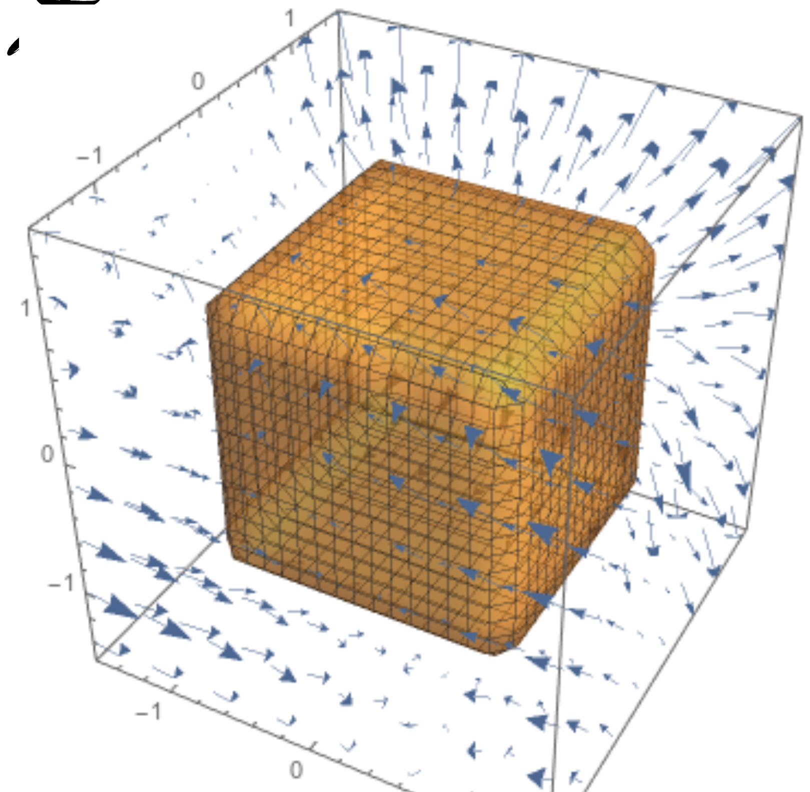


$$\int_S \vec{F}\cdot dA = \int_W \nabla\cdot\vec{F}\,dV = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(1+1+1)\,dz\,dy\,dx = 24$$

$$\vec{G}(x,y,z) = xy\,\vec{i} + yz\,\vec{j} + xyz\,\vec{k}$$

$$\int_S \vec{G}\cdot dA = \int_W \nabla\cdot\vec{F}\,dV = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(y+z+xy)\,dz\,dy\,dx$$

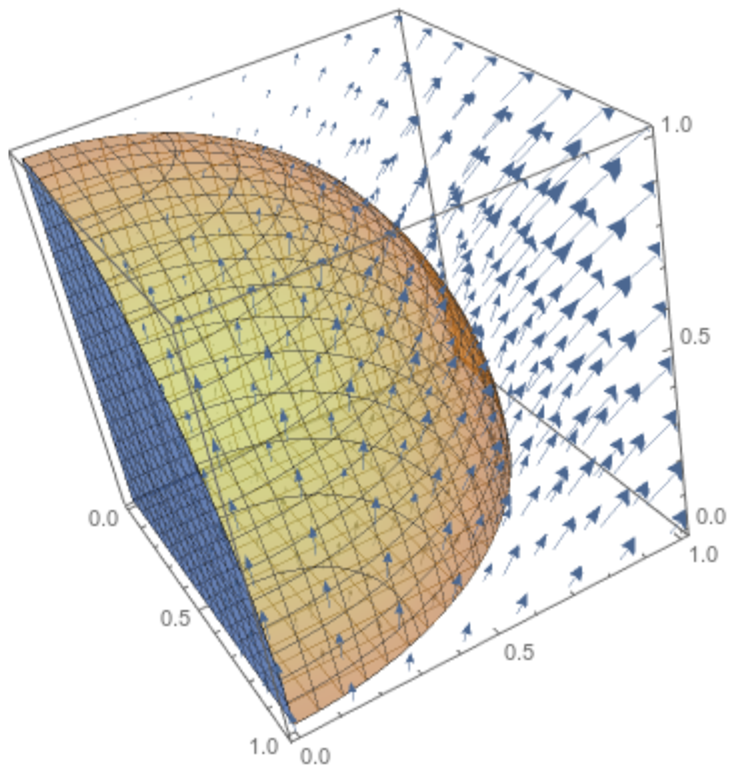
$$= 0.$$



Ex: $\int_S \vec{F}\cdot d\vec{A}$ $\quad \vec{F} = (x^2yz + y^2z)\vec{i} + (xy^2z)\vec{j} + (x^2+y^2)\vec{k}$

S is unit sphere in $x, y, z \geq 0$ octant

S has 4 pieces



$\int_S \vec{F}\cdot d\vec{A} = \int_W \nabla\cdot\vec{F}\, dV$

$= \int_W (2xyz + 2xyz + 0)\, dV$

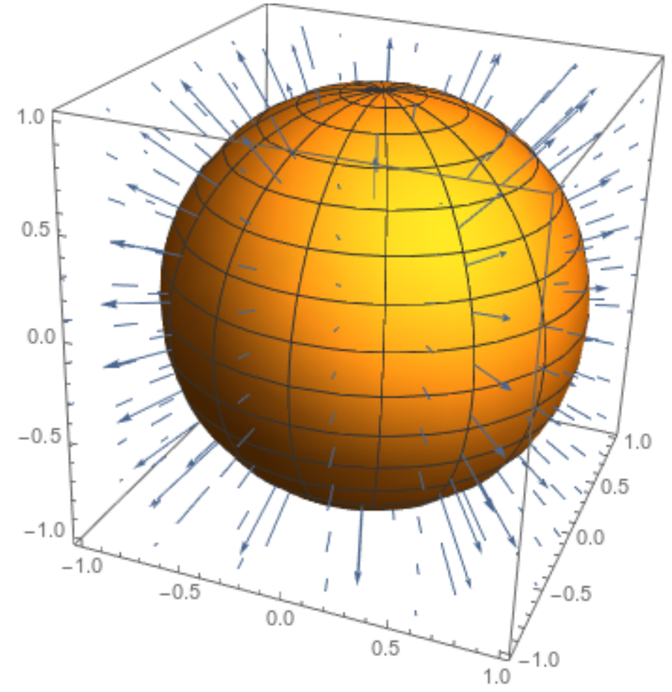
$= \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} 4(\rho\cos\theta\sin\phi)(\rho\sin\theta\sin\phi)(\rho\cos\phi)\rho^2\sin\phi \, d\phi\, d\theta\, d\rho$

$= \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} 4\rho^4 \cos\theta\sin\theta\sin^3\phi\cos\phi \, d\phi\, d\theta\, d\rho = 1/10$

Prop: If $\vec{F}$ is solenoidal on W

$S = \partial W$, then $\int_S \vec{F} \cdot d\vec{A} = \int_W \nabla \cdot \vec{F}\, dV = \int_W 0\, dV = 0.$

Ex: $\vec{F}(\vec{r}) = \frac{\vec{r}}{\|\vec{r}\|^3}$ find flux through ellipsoid $x^2 + 4y^2 + 9z^2 = 25.$



solenoidal except at O.

Easier Q: flux through unit sphere.

$\int_T \vec{F} \cdot d\vec{A} = \|\vec{F}\| \cdot SA = 1 \cdot 4\pi = 4\pi.$

Let W be region btwn T and S.

Then $\int_W \nabla \cdot \vec{F}\, dV = 0$

So $\int_{S-T} \vec{F}\, dA = 0$

so flux through ellipsoid is flux through sphere $= 4\pi.$

§9.3 Three Theorems

Three derivatives

∇ : scalar $\to$ VF

$\nabla\times$: VF $\to$ VF

$\nabla\cdot$: VF $\to$ scalar

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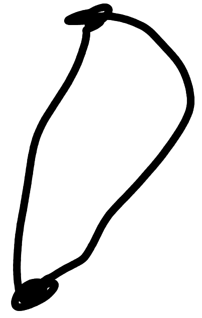
1) $\nabla\times\nabla f = \vec{0}$

If $\nabla\times\vec{F} = \vec{0}$, $F = \nabla f$

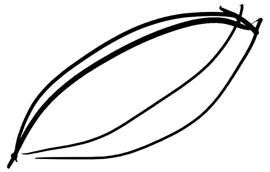
2) $\nabla\cdot\nabla\times G = \vec{0}$

If $\nabla\cdot\vec{F} = 0$, $\vec{F} = \nabla\times\vec{G}$

$$\int_C \nabla f\cdot d\vec{r} = f(Q) - f(P)$$



$$\int_S \nabla\times\vec{F}\cdot d\vec{A} = \int_C \vec{F}\cdot d\vec{r}$$



$$\int_V \nabla\cdot\vec{F}\, dV = \int_S \vec{F}\cdot d\vec{A}$$