1 Limits

1.1 What is a limit?

Example 1.1. What value "should" the function given in the following graph have at $x = 1$?

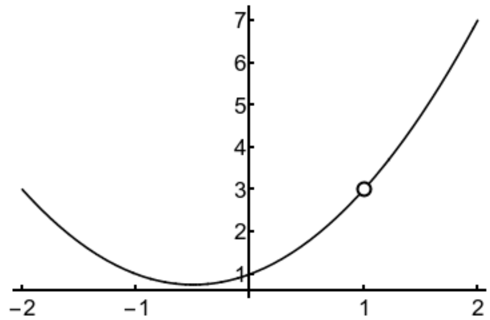

Figure 1.1: The graph of $\frac{x^3-1}{x-1}$

In this section we will study the idea of "limits". Recall a function takes an input and gives an output. The core idea of a limit is to look at the outputs for inputs "near" a given input, in order to answer the question, essentially, of what an output "should be" when we don't have a good one..

most of you have seen an informal characterization of limits before

Definition 1.2 (informal)**.** Suppose a is a real number, and f is a function which is defined for all x "near" the number a. We say "The *limit* of $f(x)$ as x approaches a is L," and we write

$$\lim_{x \to a} f(x) = L,$$

if we can make $f(x)$ get as close as we want to L by picking x that are very close to a.

Graphically, this means that if the x coordinate is near a then the y coordinate is near L. Pictorially, if you draw a small enough circle around the point $(a, 0)$ on the x-axis and look at the points of the graph above and below it, you can force all those points to be close to L.

Remark 1.3. We specifically do not consider the value of f at a when talking about limits. Limits were invented to deal with times when either a isn't in the domain of f, or when $f(a) \neq \lim_{x \to a} f(x)$.

Example 1.4. 1. If $f(x) = 3x$ then $\lim_{x \to 1} f(x) = 3$.

2. If $f(x) = x^2$ then $\lim_{x\to 0} f(x) = 0$.
3. If $f(x) = \frac{x^2-1}{x-1}$ then $\lim_{x\to 1} f(x) = 2$.

We'd like to take this definition and translate it into mathematical language, making it more precise at the same time. First we need to be a bit more precise about what "distance" means.

1.2 Distance and Absolute Value

The absolute value function $|x|$ will be important to us. It is even. We can think of it as a piecewise function

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

Or we can think of it as $|x| = \sqrt{x^2}$; both of these definitions can be useful.

The absolute value function is important because it is used to think about distances, which will be a core idea in this course.

Definition 1.5. The *distance* between two numbers x and y is the absolute value of their difference $|x - y|$.

Example 1.6. The distance between 3 and 5 is $|3 - 5| = 2$. (Notice this is the same as the distance $|5 - 3| = 2$ between 5 and 3).

The distance between 2 and -2 is $|2 - (-2)| = 4$. The distance between -2 and 2 is $|-2-2| = 4$.

We know the shortest distance between two points is a straight line; this implies that the distance between x and y is no larger than the distance from x to z plus the distance from z to y–that is, we can never make our path shorter by detouring through another point z. This leads us to formulate a very important rule known as the *Triangle Inequality*.

Fact 1.7 (Triangle Inequality)**.** *If x and y are real numbers, then $|x + y| \leq |x| + |y|$.*

Geometrically, we see that $|x + y| = |x - (-y)|$ is the distance between x and $-y$. But $|x| = |x - 0|$ is the distance between x and 0, and $|y| = |0 - (-y)|$ is the distance between 0 and $-y$. Thus, since the distance from x to $-y$ is no bigger than the distance from x to 0 plus the distance from 0 to $-y$, we have $|x + y| \leq |x| + |y|$.

We can also interpret this statement algebraically. Algebraically, we see that the x and y can cancel each other out *inside* the absolute value signs, but not *outside* them. (That is, $|5 + -3| = 2$ but $|5| + |-3| = 8$).

Example 1.8. $8 = |3+5| \leq |3| + |5| = 8$.

$4 = |6-2| \leq |6| + |-2| = 8$. (Note that we're treating $6-2$ as $6+(-2)$ here).

Example 1.9. If $|f(x)| \leq 2$ and $|g(x)| \leq |x+1|$ then $|f(x)+g(x)| \leq |f(x)| + |g(x)| \leq 2 + |x+1|$.

In contrast, if $|f(x)| \geq 2$ and $|g(x) \leq x^2$ then we can say $|f(x)+g(x)| \leq |f(x)|+|g(x)| \leq |f(x)| + x^2$. We can't get rid of the $|f(x)|$ term because we don't have an *upper* bound on it, only a lower bound.

Remark 1.10. We call this the "Triangle Inequality" because, when generalized to two dimensions, it tells us that the length of one side of a triangle is no longer than the sum of the lengths of the other two sides. In this one-dimensional case, the triangle is extremely narrow and has all three points on the same line, which makes it look rather less triangular.

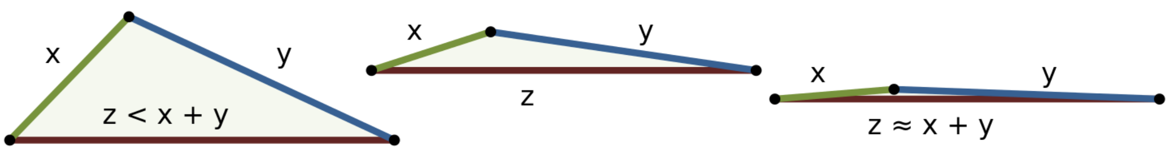

Figure 1.2: The triangle equality in two dimensions, shrinking towards one.
Image by White Timberwolf and Brews ohare licensed under CC BY-SA 3.0.

We can also derive the *inverse triangle inequality*:

Lemma 1.11. *If x and y are real numbers, then $|x+y| \geq |x| - |y|$.*

Proof. The algebraic proof is short, and takes advantage of the fact that $|y| = |-y|$ for any number y.

$$|x| = |x+y-y| \leq |x+y| + |-y| \qquad \text{by the triangle inequality}$$
$$|x| - |y| \leq |x+y| + |-y| - |y| = |x+y|.$$

□

Geometrically, this tells us that any side of a triangle is longer than the difference of the lengths of the other two sides–which should make sense, since it needs to be long enough to connect them.

Example 1.12. $8 = |3+5| \geq |3| - |5| = -2$. Also, $8 = |5+3| \geq |5| - |3| = 2$.

$4 = |6-2| \geq |6| - |-2| = 4$. Also, $4 = |2-6| \geq |2| - |-6| = -4$.

Example 1.13. If $|f(x)| \geq 2$ and $|g(x)| \leq |x+1|$ then $|f(x)+g(x)| \geq |f(x)| - |g(x)| \geq 2 - |x+1|$.

In contrast, if $|f(x)| \leq 2$ and $|g(x) \leq x^2$ then we can say $|f(x)+g(x)| \geq |f(x)| - |g(x)| \geq |f(x)| - x^2$. We can't get rid of the $|f(x)|$ term because we don't have a *lower* bound on it, only an upper bound.

Example 1.14. Suppose $|f(x)| \leq 3$ and $|g(x)| \leq x^2$. What can we say about $|f(x)+g(x)|$ and $|f(x) - g(x)|$?

$|f(x)+g(x)| \leq |f(x)| + |g(x)| \leq 3 + x^2$ (by the triangle inequality).

$|f(x) - g(x)| \geq |f(x)| - |g(x) \geq |f(x)| - x^2$ (by the reverse triangle inequality). We can't say anything more about the $|f(x)|$ bit because knowing that $|f(x)|$ is *smaller* than something doesn't tell us anything about what it's *bigger* than.

Remark 1.15. The important difference between the triangle inequality and the reverse triangle inequality isn't the starting point; they both start with $|x+y|$. But the triangle inequality tells us that $|x+y|$ is smaller than something, while the inverse triangle inequality tells us that $|x+y|$ is bigger than something.

1.3 The Formal Definition of a Limit

Definition 1.16. Suppose a is a real number, and f is a function defined on some open interval containing a, except possibly for at a. We say the *limit* of $f(x)$ as x approaches a is L, and write

$$\lim_{x \to a} f(x) = L,$$

if for every real number $\epsilon > 0$ there is a real number $\delta > 0$ such that whenever $0 < |x-a| < \delta$ then $|f(x) - L| < \epsilon$.

Importantly, you should notice that this is *exactly the same thing we said before!* ϵ represents "how close we want $f(x)$ to get to L" and δ represents "how close x needs to get to a".

Then this definition says that if we pick any margin of error $\epsilon > 0$, then there is some distance δ such that if x is within distance δ of a, then $f(x)$ is within our margin of error ϵ of L.

Remark 1.17. The Greek letter epsilon (ϵ) became the letter "e", and stands for "error". The Greek letter delta (δ) became the letter "d", and stands for "distance". This isn't just a mnemonic for you; this is actually why those letters were chosen.

Example 1.18. 1. If $f(x) = 3x$ then prove $\lim_{x\to 1} f(x) = 3$.

Let $\epsilon > 0$ and set $\delta = \underline{\epsilon/3}$. Then if $|x-1| < \delta$ then

$$|f(x) - 3| = |3x - 3| = 3|x-1| < 3\delta = \epsilon.$$

2. If $f(x) = x^2$ then prove $\lim_{x\to 0} f(x) = 0$.

Let $\epsilon > 0$ and set $\delta = \underline{\sqrt{\epsilon}}$. Then if $|x-0| < \delta$, then

$$|f(x) - 0| = |x^2| = |x|^2 < (\sqrt{\epsilon})^2 = \epsilon.$$

3. If $f(x) = \frac{x^2-1}{x-1}$ then $\lim_{x\to 1} f(x) = 2$.

This is harder to see at first, until we recall or notice that this function is mostly the same as $x+1$.

Let $\epsilon > 0$ and let $\delta = \underline{\epsilon}$. Then if $0 < |x-1| < \delta$, we have

$$\begin{aligned} |f(x) - 2| &= \left| \frac{x^2-1}{x-1} - 2 \right| \\ &= |x+1-2| && \text{since } x \neq 1 \\ &= |x-1| < \delta = \epsilon. \end{aligned}$$

Remark 1.19. Despite the fact that we set δ as the first thing we do in the proof, we often figure out what it should be last. I strongly recommend beginning your proof by writing "And set $\delta =$ ____" and then working out the proof. By the time you get to the end you'll know what δ needs to be and you can go back and fill in th blank.

Example 1.20. If $f(x) = 4x - 2$ then find (with proof!) $\lim_{x\to -2} f(x)$.

We first need to generate a "guess". This is a nice function, so it seems like the answer should be close to $f(-2) = -10$.

Let $\epsilon > 0$ and set $\delta = \underline{\epsilon/4}$. Then if $|x-(-2)| < \delta$ we compute

$$|f(x) + 10| = |4x - 2 + 10| = |4x + 8| = 4|x+2| < 4\delta = \epsilon.$$

Example 1.21. If $f(x) = x^2$ find (with proof!) $\lim_{x\to 3} f(x)$.

We first need to generate a "guess". This is a nice function, so it seems like the answer should be close to $f(3) = 9$.

Let $\epsilon > 0$ and set $\delta \leq \underline{\epsilon/7, 1}$. Then if $|x-3| < \delta$ we compute

$$|x^2 - 9| = |x+3| \cdot |x-3| < |x+3|\delta$$

but this is kind of a problem because we still have an x floating around. But logically, we know that if δ is small enough, x will be close to 3 and thus $|x+3|$ will be close to 6.

To guarantee that $|x+3|$ is actually close to 6, we'll require $\delta \leq 1$ as well. Then we compute

$$\begin{aligned} |x^2-9| &< |x+3|\delta = |(x-3)+6| \cdot \delta \\ &\leq (|x-3| + |6|)\,\delta && \text{by the triangle inequality} \\ &< (1+6)\delta = 7\delta. \end{aligned}$$

Notice we said that $|x+3|$ would be close to 6, and what we actually showed is that $|x+3| \leq 7$–which of course it is if it is close to 6.

So now we just need to make sure δ is small enough that $7\delta \leq \epsilon$, so in addition to letting $\delta \leq 1$ we also let $\delta \leq \epsilon/7$, so we have

$$|x^2-9| < 7\delta = 7\epsilon/7 = \epsilon.$$

Remark 1.22.

- We often use an approach of isolating all our xs and turning them into an $x-3$ or $x-a$ or whatever we *know how to control*. Since in example 1.21 we know that $|x-3| < \delta$ we want to turn all our xs into $|x-3|$s. Then we can deal with whatever is left over.
- Notice that here we didn't actually say what δ is; we just listed some properties it needs to have, by saying that $\delta \leq \epsilon/12, 1$. If we want to pick out a specific number, we can write $\delta = \min(\epsilon/12, 1)$, but this isn't actually necessary.

Example 1.23. If $f(x) = x^2 + x$, find (with proof) $\lim_{x\to 2} f(x)$.

This is a nice function, so it seems like the answer should be close to $f(2) = 6$.

Let $\epsilon > 0$ and set $\delta < \underline{\sqrt{\epsilon/2}, \epsilon/10}$. Then if $|x-2| < \delta$ we have

$$\begin{aligned} |f(x)-6| &= |x^2+x-6| = |(x^2-4)+(x-2)| \\ &\leq |x^2-4| + |x-2| && \text{(triangle inequality)} \\ &= |x-2| \cdot |x+2| + |x-2| = |x-2|\,(|x+2|+1) \\ &= |x-2|\,(|x-2+4|+1) \leq |x-2|\,(|x-2|+5) && \text{(triangle inequality)} \\ &< \delta(\delta+5) = \delta^2 + 5\delta. \end{aligned}$$

You could try to figure out exactly when $\delta^2 + 5\delta = \epsilon$, and after some quadratic formula-ing you'd find you need $\delta \leq \frac{-5+\sqrt{25+4\epsilon}}{2}$. But that's tedious and actually way too much work. (But if you prefer this approach it's perfectly acceptable).

It's easier to instead list two conditions: we let $\delta \leq \sqrt{\epsilon/2}, \epsilon/10$. Then $\delta^2 \leq \epsilon/2$ and $5\delta \leq \epsilon/2$, and we have

$$|f(x) - 6| < \delta^2 + 5\delta \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Example 1.24. If $f(x) = 1/x$, find (with proof) $\lim_{x \to 4} f(x)$.

Since $f(x)$ is a nice function, we guess $f(4) = 1/4$.

Let $\epsilon > 0$ and set $\delta < \underline{1, 12\epsilon}$. Then we compute

$$|f(x) - 1/4| = |1/x - 1/4| = \left| \frac{4-x}{4x} \right| = \frac{|x-4|}{|4x|} < \frac{\delta}{|4x|}$$

and we need to do something about the x on the bottom. In this case we need to ensure that $|4x|$ is *big* enough since we're dividing by it. We see that $|4x| = |4(x-4+4)| = |4(x-4)+16|$; how can we make this bigger than something?

Here we use the inverse triangle inequality, after a bit of rewriting. We compute

$$\begin{aligned}
|4(x-4) + 16| = |16 - 4(4-x)| &\geq |16| - |4(4-x)| = 16 - 4|x-4| \\
|x-4| &< \delta < 1 \\
-4|x-4| &> -4 \\
16 - 4|x-4| &> 12.
\end{aligned}$$

Now we can compute

$$|1/x - 1/4| < \frac{\delta}{|4x|} < \frac{12\epsilon}{12} = \epsilon.$$

Example 1.25. If $f(x) = \frac{x-1}{x^2-1}$ then find (with proof!) $\lim_{x \to 1} f(x)$?

We notice that if $x \neq 1$, then $f(x) = \frac{1}{x+1}$, and so we guess $\lim_{x \to 1} f(x) = 1/2$.

Let $\epsilon > 0$ and let $\delta = \underline{\epsilon, 1}$. Then if $|x-1| < \delta$ we have

$$\begin{aligned}
|f(x) - 1/2| &= \left| \frac{x-1}{x^2-1} - 1/2 \right| = \left| \frac{1}{x+1} - 1/2 \right| \qquad \text{because } x \neq -1 \\
&= \left| \frac{2}{2x+2} - \frac{x+1}{2x+2} \right| = \left| \frac{1-x}{2x+2} \right| \\
&= \frac{|x-1|}{|2(x-1)+4|}.
\end{aligned}$$

We want to make the top small, so require $\delta < \epsilon$. We want the bottom to be big, say we want it to be at least two. We see that

$$|2(x-1) + 4| = |4 - 2(1-x)| \geq 4 - 2|1-x| > 4 - 2\delta$$

so if we require $\delta < 1$ this gives us

$$|2(x-1)+4| > 2.$$

Thus we have

$$|f(x) - 1/2| = \frac{|x-1|}{|2x+2|} < \frac{\delta}{4-2\delta} < \delta/2 < \epsilon/2 < \epsilon.$$

Example 1.26. Now suppose

$$f(x) = \begin{cases} \frac{x-1}{x^2-1} & x \neq 1 \\ 2 & x = 1 \end{cases}$$

What is $\lim_{x \to 1} f(x)$?

This looks really nasty, but is actually easy after we already did Example 1.25.

The limit doesn't care about what happens at any one specific point, and especially doesn't care about what happens at 1. So for our purposes, this function is the same as $f(x) = \frac{x-1}{x^2-1}$, and thus the limit is, as before, 1/2.

Let $\epsilon > 0$, and let $\delta < \epsilon, 1$. Then

$$|f(x) - 1/2| = \left| \frac{x-1}{x^2-1} - 1/2 \right| < \epsilon$$

as computed in Example 1.25. (This is a completely valid proof as written!)