2 More on Limits

2.1 When Limits Don't Exist

In the last section we proved that a bunch of limits exist. Now we'll look at some functions and limits that don't behave so nicely.

Example 2.1. The *Heaviside Function* or *step function* is given by

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

What is $\lim_{x \to 0} H(x)$?

Looking at the graph it seems like no limit exists; when x is close to zero, sometimes $H(x)$ is 0 and sometimes $H(x)$ is 1, and you can't get "close enough" to make that stop.

We want to prove that no limit exists, so we have to look at our definition–and do the exact opposite of what we do to prove a limit does exist. Normally we want to say that for *any* expected error, we can get close enough to be within that error. So we say we can start with any ϵ, and then find a good enough δ.

In this situation we want to say there is some error we *cannot* hit. So we start by picking some ϵ, and then proving that no δ will work.

(I like to think of this as a game. When I say a limit exists, I'm telling you you can pick *any* ϵ, and I can find a δ. In contrast, here I'm saying that you have a winning move–there's some ϵ you can pick where I can't find a δ).

Proof. Suppose the limit exists–that is there is some number L such that for ever $\epsilon > 0$, there is a $\delta > 0$ such that if $|x - 0| < \delta$ then $|H(x) - L| < \epsilon$.

Fix $\epsilon = 1/2$. (We will see why I picked this specific value in a bit). Then let $\delta > 0$ be any (positive) real number.

Let $x_1 = \delta/2$. Then $|x_1 - 0| = \delta/2 < \delta$. Thus by definition of limit,

$$\epsilon > |H(x_1) - L| = |1 - L|.$$

Thus $|1 - L| < 1/2$.

Now let $x_2 = -\delta/2$. Then $|x_2 - 0| = \delta/2 < \delta$, and by definition of limit,

$$\epsilon > |H(x_2) - L| = |0 - L| = |L|.$$

Thus $|L| < 1/2$.

So what does this mean? Since $|1 - L| < 1/2$ we know that L is within $1/2$ of 1. Since $|L| < 1/2$ we know that L is within $1/2$ of 0. But there are no numbers that are within $1/2$ of both 0 and 1, so L cannot exist!

We can translate this into more mathematical language in two different ways.

We can add our two inequalities, to get $|1 - L| + |L| < 1/2 + 1/2$. But if we look at the left hand side of that, that looks like part of the triangle inequality. So we have

$$1 > |1 - L| + |L| \geq |1 - L + L| = |1|$$

but this is false and thus we have a contradiction. So no such L can exist, and the limit does not exist.

Alternatively, we can use the inverse triangle inequality, which tells us that $1/2 > |1-L| \geq 1 - |L|$. Adding our two inequalities together now gives $1 - |L| + |L| < 1/2 + 1/2$ and thus $1 < 1$, which is a contradiction. So no such L exists, and the limit does not exist.

(Notice that all three of these arguments are essentially the same!)

□

Example 2.2. What is $\lim_{x\to 1} H(x)$?

There's nothing funny going on here, it looks like the limit should be 1. And indeed proving this in this case is quite easy.

Let $\epsilon > 0$ and let $\delta = \underline{1}$. Then if $|x - 1| < \delta$ then in particular we have $|x - 1| < 1$ and thus $x - 1 > -1$ so $x > 0$ and $H(x) = 1$. Then we have$|H(x) - 1| = 0 < \epsilon$.

Example 2.3. Let

$$f(x) = \begin{cases} x & x < 1 \\ x + 2 & x > 1 \end{cases}$$

What is $\lim_{x\to 3} f(x)$?

We guess 5. Let $\epsilon > 0$ and set $\delta \leq \underline{2, \epsilon}$. Then if $0 < |x - 3| < \delta$, then we see that in particular $|x - 3| < 2$. This implies that $x - 3 > -2$ and thus $x > 1$, so $f(x) = x + 2$. Then

$$|f(x) - 5| = |x + 2 - 5| = |x - 3| < \delta \leq \epsilon.$$

Thus $\lim_{x\to 3} f(x) = 5$.

Example 2.4. Now show that $\lim_{x\to 1} f(x)$ does not exist.

Suppose (for contradiction) that $\lim_{x\to 1} f(x) = L$. Then set $\epsilon = \underline{1}$ and let $\delta > 0$. Pick $x_1 = 1 + \delta/2$ and $x_2 = 1 - \delta/2$.

Then $|x_1 - 1| = \delta/2 < \delta$, so we know that

$$\epsilon > |f(x_1) - L| = |x_1 + 2 - L| = |3 + \delta/2 - L|.$$

Similarly, $|x_2 - 1| = \delta/2 < \delta$ so we know that

$$\epsilon > |f(x_2) - L| = |x_2 - L| = |1 - \delta/2 - L| = |L + \delta/2 - 1|.$$

Adding these two equations gives

$$\begin{aligned} 2\epsilon &> |L + \delta/2 - 1| + |3 + \delta/2 - L| \\ &\geq |L + \delta/2 - 1 + 3 + \delta/2 - L| = |2 + \delta| \\ &\geq 2 + |\delta| > 2. \end{aligned}$$

But since $\epsilon = 1$ this gives us $2 > 2$ which is a contradiction. Thus no such limit exists.

Example 2.5.

$$g(x) = \begin{cases} 4x - 2 & x < -2 \\ 2x + 5 & x > -2 \end{cases}$$

What is $\lim_{x \to -2} g(x)$?

We claim the limit does not exist.

Suppose (for contradiction) that $\lim_{x \to -2} g(x) = L$. Then set $\epsilon =$ and let $\delta > 0$. Pick $x_1 = -2 + \delta/2$ and $x_2 = -2 - \delta/2$.

Then $|x_1 + 2| = \delta/2 < \delta$, so we know that

$$\epsilon > |g(x_1) - L| = |2(-2 + \delta/2) + 5 - L| = |-4 + \delta + 5 - L| = |1 + \delta - L|.$$

Similarly, $|x_2 - 1| = \delta/2 < \delta$ so we know that

$$\epsilon > |g(x_2) - L| = |4(-2 - \delta/2) - 2 - L| = |-8 - 2\delta - 2 - L| = |-10 - 2\delta - L| = |L + 2\delta + 10|.$$

Adding these two equations gives

$$\begin{aligned} 2\epsilon &> |L + 2\delta + 10| + |1 + \delta - L| \\ &\geq |L + 2\delta + 10 + 1 + \delta - L| = |11 + 3\delta| \\ &\geq 11 + 3|\delta| > 11. \end{aligned}$$

But since $\epsilon = 1$ this gives us $2 > 11$ which is a contradiction. Thus no such limit exists.

Example 2.6. What is $\lim_{x\to 0} \sin(1/x)$?

If we look at a graph of the function, it's hard to see what the limit could be–the function jumps up and down crazily near 0, and it doesn't look like you can get "close enough" to avoid this.

Suppose there is some L with $\lim_{x\to 0} f(x) = L$. Let $\epsilon = \underline{1}$ and fix some $\delta > 0$. Write $f(x) = \sin(1/x)$. I claim there is some positive integer n such that $0 < \frac{2}{n\pi} < \delta$ (pick a number such that $n > \frac{2}{\pi\delta}$.) In fact, we can pick n so that $\sin(n\pi/2) = 1$.

Let $x_1 = \frac{2}{n\pi}$. Then $|x_1 - 0| < \delta$ by construction, and

$$|f(x_1) - L| = |\sin(n\pi/2) - L| = |1 - L| < \epsilon = 1.$$

Let $x_2 = \frac{-2}{n\pi}$. Then $|x_2 - 0| < \delta$, and

$$|f(x_2) - L| = |\sin(-n\pi/2) - L| = |-1 - L| < \epsilon = 1.$$

Informally: L must be within 1 of both the number 1 and the number -1; no number satisfies these conditions, so the limit cannot exist.

Formally: there are two ways to see this. One is to notice that $|1 + L| + |1 - L|$ looks like it comes from the triangle inequality. So we have

$$1 + 1 > |1 + L| + |1 - L| \geq |1 + L + 1 - L| = 2$$

which is a contradiction. Thus no such L exists.

The other way is to decide things will be easier if we can make all the Ls be in the same form. We have a $|1 - L| = |L - 1|$ and we can rewrite $|1 + L| = |L + 2 - 1| \leq |L - 1| + 2$. Now we can add our two inequalities together, and we get

$$2 = 1 + 1 > |1 - L| + |1 + L| \geq |1 - L| + |1 - L| + 2 \geq 2$$

which is impossible. Thus no such L exists.

A summary of the layout of these proofs: Assume a limit $\lim_{x\to a} f(x)$ exists. Pick a value for ϵ (you can leave this blank until later), and then let $\delta > 0$ be some real number.

Now find two points that are both close to your limit point a, but give very different outputs. Use the assumption that the limit exists to see that $\epsilon > |f(x_1) - L|$ and $\epsilon > |f(x_2) - L|$ and add these together; use the triangle inequality to cancel the Ls, and get $2\epsilon > f(x_1) - f(x_2)$.

Hopefully the right hand side is a constant (or bigger than a constant), and we can pick ϵ to be small enough that this can't work.

Remark 2.7. In most of the cases we've looked at, we could say that the *one-sided limits* exist, by looking at what happens only when we approach from one side or the other. We can build a theory of one-sided limits that looks very much like the theory we just built; however, I won't be spending time on it in this course. There are some notes about it at the end of this section, but the material won't be covered in this course.

2.2 Infinite limits

In this section we'll talk about limits that involve infinity. Some limits deal with infinity as an output, and others deal with it as an input (or both).

Remark 2.8. Recall that infinity is not a number. Sometimes while dealing with infinite limits we might make statements that appear to treat infinity as a number. But it's not safe to treat ∞ like a true number and we will be careful of this fact.

Definition 2.9. We write

$$\lim_{x \to a} f(x) = +\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily large (and positive).

We write

$$\lim_{x \to a} f(x) = -\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily negative.

We write

$$\lim_{x \to a} f(x) = \pm\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily positive or negative. We usually use this when both occur.

Formally:

Definition 2.10. We write

$$\lim_{x \to a} f(x) = +\infty$$

if for every $N > 0$ there is a $\delta > 0$ so that if $0 < |x - a| < \delta$ then $f(x) > N$.

We write

$$\lim_{x \to a} f(x) = -\infty$$

if for every $N > 0$ there is a $\delta > 0$ so that if $0 < |x - a| < \delta$ then $f(x) < -N$.

We write

$$\lim_{x \to a} f(x) = \pm\infty$$

if for every $N > 0$ there is a $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x)| > N$.

Remark 2.11. Important note: If the limit of a function is infinity, the limit *does not exist*. This is utterly terrible English but I didn't make it up so I can't fix it. Most of the theorems that say "If a limit exists" are not including cases where the limit is infinite.

Also notice that if a limit is $+\infty$ or is $-\infty$, then it is also true to say that the limit is $\pm\infty$. Remember that ∞ isn't a number; the statement that a limit is $+\infty$ and that it's $\pm\infty$ are not mutually contradictory, and in fact the first implies the second. And both statements imply that the limit does not exist.

Thus it would sometimes be correct to say that $\lim_{x\to a} f(x) = +\infty$, or to say that $\lim_{x\to a} f(x) = \pm\infty$, or to say that $\lim_{x\to a} f(x)$ does not exist, all about the same function at the same time.

On a test you should always try to make the most precise statement you can. If a limit is $+\infty$ and you say that the limit does not exist, you are *correct* but you will not get full credit.

Example 2.12. The most important example, that we do all our work in reference to, is that $\lim_{x\to 0} 1/x = \pm\infty$. In particular, if x is very small and positive our output will be very large and positive; if x is very small and negative then our output will be very large and negative.

Fix some $N > 0$, and set $\delta = \underline{1/N}$. Then if $0 < |x - 0| < \delta$ then

$$|1/x| = 1/|x| > 1/\delta = 1/(1/N) = N.$$

Example 2.13. $\lim_{x\to 0} 1/x^2 = +\infty$. We can see that the values will get very large because the limit of the top is a constant and the limit of the bottom is 0; but both the numerator and the denominator will always be positive, since $x^2 \geq 0$ (and of course $1 \geq 0$).

Formally: fix some $N > 0$ and set $\delta = \underline{1/\sqrt{N}}$. Then if $0 < |x - 0| < \delta$ then

$$1/x^2 > 1/\delta^2 = 1/(1/\sqrt{N})^2 = N.$$

Thus $\lim_{x\to 0} 1/x^2 = +\infty$.

Notice that we didn't use absolute values in this proof—this is the difference between a limit being $+\infty$ and a limit being $\pm\infty$.

Notice also that we easily *could* prove that $\lim_{x\to 0} 1/x^2 = \pm\infty$.

Example 2.14. $\lim_{x\to 1} \frac{x}{x-1} = \pm\infty$. We see that the top will approach 1 and the bottom will approach 0 so the limit is $\pm\infty$. The top is always positive near 0; the bottom is positive to the right and negative to the left, so the sign flips. Thus the answer is $\pm\infty$.

Formally: Let $N > 0$ and set $\delta \leq \underline{1/2, 1/2N}$. Then if $0 < |x-1| < \delta$, we compute

$$\left|\frac{x}{x-1}\right| = \frac{|x-1+1|}{|x-1|} \geq \frac{1-|x-1|}{|x-1|}$$
$$\geq \frac{1-\delta}{|x-1|} > \frac{1/2}{\delta} = \frac{1/2}{1/2N} = N.$$

Thus $\lim_{x\to 1} \frac{x}{x-1} = \pm\infty$.

Note that we can't be more precise, since the outputs of f get both very positive and very negative in a small window around 1. If we wanted to prove this, we can look at the one-sided limits.

Example 2.15. Compute with proof $\lim_{x\to -3} \frac{x}{(x+3)^2}$.

Let $N > 0$ and set $\delta \leq \underline{2,}$. Then if $0 < |x+3| < \delta$, we compute

$$\frac{x}{(x+3)^2} = \frac{x+3-3}{(x+3)^2} \leq \frac{|x+3|-3}{(x+3)^2} < \frac{\delta-3}{(x+3)^2}$$
$$< \frac{-1}{(x+3)^2} < \frac{-1}{\delta^2} \leq \frac{-1}{(1/\sqrt{N})^2}$$
$$= \frac{-1}{1/N} = -N.$$

Alternatively, we could notice that since $|x+3| < \delta$, we have $-\delta < x+3 < \delta$ and thus $-\delta - 3 < x < \delta - 3$, and get

$$\frac{x}{(x+3)^2} < \frac{-3+\delta}{(x+3)^2} \leq \frac{-1}{(x+3)^2}$$
$$< \frac{-1}{\delta^2} \leq \frac{-1}{(1/\sqrt{N})^2}$$
$$= \frac{-1}{1/N} = -N.$$

Thus $\lim_{x\to -3} \frac{x}{(x+3)^2} = -\infty$.

Example 2.16. Compute with proof $\lim_{x\to 4} \frac{-1}{(x-4)^2}$.

We estimate that the limit is $+\infty$. Let $N > 0$ and set $\delta \leq \underline{1/\sqrt{N}}$. Then if $0 < |x-4| < \delta$, we know that

$$\frac{1}{|x-4|} > \frac{1}{\delta} \geq \sqrt{N}$$
$$\frac{1}{(x-4)^2} > \frac{1}{\delta^2} \geq N$$
$$\frac{-1}{(x-4)^2} < \frac{-1}{\delta^2} \leq -N.$$

Thus by definition, $\lim x \to 4 \frac{-1}{(x-4)^2} = -\infty$.

2.3 Limits at infinity

We now ask what happens to a function as the input gets arbitrarily large (or negative).

Definition 2.17. Let f be a function defined for (a, ∞) for some number a. We write

$$\lim_{x \to +\infty} f(x) = L$$

to indicate that when x is large enough, the values of $f(x)$ get arbitrarily close to L. Formally, if for every $\epsilon > 0$ there is a $M > 0$ so that if $x > M$ then $|f(x) - L| < \epsilon$.

We write

$$\lim_{x \to +\infty} f(x) = \pm\infty$$

to indicate that when x is large enough, $f(x)$ is also very large or very negative. Formally, if for every $N > 0$ there is a $M > 0$ so that if $x > M$ then $|f(x)| > N$.

Let f be a function defined for $(-\infty, a)$ for some number a. We write

$$\lim_{x \to -\infty} f(x) = L$$

to indicate that when x is negative enough, the values of $f(x)$ get arbitrarily close to L. Formally, if for every $\epsilon > 0$ there is a $M > 0$ so that if $x < -M$ then $|f(x) - L| < \epsilon$.

We write

$$\lim_{x \to -\infty} f(x) = \pm\infty$$

to indicate that when x is negative enough, $f(x)$ is also very large or very negative. Formally, if for every $N > 0$ there is a $M > 0$ so that if $x < -M$ then $|f(x)| > N$.

If the limit at infinity is finite we say that the curve $y = f(x)$ has a *horizontal asymptote* at the line $y = L$. If the limit at infinity is infinite we will sometimes talk about a *slant asymptote*.

Example 2.18. The most important limits to calculate, as before, are the limits of x and $1/x$. It seems reasonable that $\lim_{x \to +\infty} x = +\infty$ and $\lim_{x \to -\infty} x = -\infty$.

Let $N > 0$ and set $M = \underline{N}$. Then if $x > M$ we have $x > M = N$, so $\lim_{x \to +\infty} x = +\infty$. Similarly, if $x < -M$ then $x < -M = -N$ so $\lim_{x \to -\infty} x = -\infty$.

For $f(x) = 1/x$, we expect the limit to be 0. So let $\epsilon > 0$ and set $M = \underline{1/\epsilon}$. Then if $x > M$, we have

$$|f(x) - 0| = 1/|x| < 1/M = 1/(1/\epsilon) = \epsilon.$$

Similarly, if $x < -M$ then

$$|f(x) - 0| = 1/|x| < 1/|-M| = 1/M = \epsilon.$$

Example 2.19. What is $\lim_{x\to+\infty} 1/x^2$?

We can compute this directly or indirectly. Directly: Let $\epsilon > 0$ and let $N = 1/\sqrt{\epsilon}$. Then if $x > N$ we have

$$\left|\frac{1}{x^2} - 0\right| = \frac{1}{x^2} < \frac{1}{N^2} = \frac{1}{(1/\sqrt{\epsilon})^2} = \epsilon.$$

Thus $\lim_{x\to+\infty} 1/x^2 = 0$.

Example 2.20. What is $\lim_{x\to+\infty} \frac{3}{x+3}$?

We expect that this is zero. So let $\epsilon > 0$ and set $M \geq \underline{3/\epsilon - 3, 1}$. Then if $x > M$ then

$$\left|\frac{3}{x+3} - 0\right| = \frac{3}{|x+3|} = \frac{3}{x+3} < \frac{3}{M+3} \leq \frac{3}{3/\epsilon - 3 + 3} = \frac{3}{3/\epsilon} = \epsilon.$$

Example 2.21. What is $\lim_{x\to-\infty} \frac{2}{x^2}$?

We expect that this is also zero. So let $\epsilon > 0$ and set $M \geq \underline{\sqrt{2/\epsilon}}$. Then if $x < -M$, we have

$$\left|\frac{2}{x^2} - 0\right| = \frac{2}{x^2} < \frac{2}{M^2} \leq \frac{2}{2/\epsilon} = \epsilon.$$

Example 2.22. What is $\lim_{x\to+\infty} \frac{2x+3}{x+5}$?

We guess 2. Let $\epsilon > 0$ and set $M \geq \underline{7/\epsilon - 5, 1}$. Then if $x > M$ we have

$$\begin{aligned}\left|\frac{2x+3}{x+5} - 2\right| &= \left|\frac{2x+3}{x+5} - \frac{2x+10}{x+5}\right| = \left|\frac{-7}{x+5}\right| \\ &= \frac{7}{x+5} \leq \frac{7}{7/\epsilon} = \epsilon.\end{aligned}$$

Example 2.23. Does $\lim_{x\to-\infty} \frac{1}{\sqrt{x}}$ exist? Why?

It doesn't because $\frac{1}{\sqrt{x}}$ is not defined for negative numbers, so we can't even approach $-\infty$.

Example 2.24. What is $\lim_{x\to+\infty} \frac{x^2}{x-3}$? We guess $+\infty$.

Let $N > 0$ and set $M \geq \underline{4, N}$. Then if $x > M$ we have

$$\frac{x^2}{x-3} > \frac{x^2}{x} = x > M \geq N.$$

Example 2.25. What is $\lim_{x\to-\infty} \frac{x^2}{x-3}$? We guess $-\infty$. (It's negative because the top will be positive and the bottom will be negative).

This is another one of those problems with hidden negative signs, which will get confusing. Let $N > 0$ and set $M \geq \underline{3, 2N}$. Then if $x < -M$ we observe that since $x - 3 \geq 2x$ (since $x < -M \leq -3$) then $\frac{1}{x-3} \leq \frac{1}{2x}$, and then we have

$$\frac{x^2}{x-3} \leq \frac{x^2}{2x} = x/2 < -M/2 \leq -N.$$

Example 2.26. What is $\lim_{x \to +\infty} \sin(x)$?

This is exactly like $\lim_{x \to 0} \sin(1/x)$. The limit does not exist for the same reason; as x gets bigger and bigger, the function $\sin(x)$ continues to oscillate and does not approach one value, nor does it increase without bound.

	$x \to a$ δ is small	$x \to +\infty$ M is large	$x \to -\infty$ M is large
$\lim f(x) = L$ ϵ is small	Let $\epsilon > 0$ and set $\delta \leq$ __. Then if $0 < \|x - a\| < \delta$ then $\|f(x) - L\| < \epsilon$.	Let $\epsilon > 0$ and set $M \geq$ __. Then if $x > M$ then $\|f(x) - L\| < \epsilon$.	Let $\epsilon > 0$ and set $M \geq$ __. Then if $x < -M$ then $\|f(x) - L\| < \epsilon$.
$\lim f(x) = \pm\infty$ N is large	Let $N > 0$ and set $\delta \leq$ __. Then if $0 < \|x - a\| < \delta$ then $\|f(x)\| > N$.	Will Not Occur	Will Not Occur
$\lim f(x) = +\infty$ N is large	Let $N > 0$ and set $\delta \leq$ __. Then if $0 < \|x - a\| < \delta$ then $f(x) > N$.	Let $N > 0$ and set $M \geq$ __. Then if $x > M$ then $f(x) > N$.	Let $N > 0$ and set $M \geq$ __. Then if $x < -M$ then $f(x) > N$.
$\lim f(x) = -\infty$ N is large	Let $N > 0$ and set $\delta \leq$ __. Then if $0 < \|x - a\| < \delta$ then $f(x) < -N$.	Let $N < 0$ and set $M \geq$ __. Then if $x > M$ then $f(x) < -N$.	Let $N > 0$ and set $M \geq$ __. Then if $x < -M$ then $f(x) < -N$.

	Input finite	input infinite
Output finite	δ, ϵ	M, ϵ
Output infinite	δ, N	M, N

Bonus material: One-sided limits

In section 2.1 we looked at various limits that don't exist; but some are much, much nicer than other. Looking at the graph, there are two plausible values you could give for $\lim_{x\to 0} H(x)$; for $\lim_{x\to 0} \sin(1/x)$ there are infinitely many. We can capture this difference with the idea of a *one-sided limit*:

Definition 2.27. Suppose a is a real number, and f is a function defined on some open interval $(a-h, a)$ We say the *limit* of $f(x)$ as x approaches a *from the left* is L, and write

$$\lim_{x\to a^-} f(x) = L,$$

if for every real number $\epsilon > 0$ there is a real number $\delta > 0$ such that whenever $a-\delta < x < a$ then $|f(x) - L| < \epsilon$.

Remark 2.28. This says that x has to be within δ of a, but also smaller than a–thus "on the left." This captures the idea that $f(x)$ gets close to L when x is sufficiently close to a but still smaller and to the left.

Thus this intuitively, this captures the idea that the outputs of a function get close to one value on one side of a point, and perhaps get close to a different value (or are simply ill-behaved) on the other side.

Notice the subscript a^- in the limit sign. Recall that we use a $-$ sign because we're looking at what happens for inputs less than a.

Remark 2.29. The $a - \delta < x < a$ might look unrelated to what we've done so far. But note that $|x - a| < \delta$ is the same as $-\delta < x - a < \delta$, which is the same as $a - \delta < x < a + \delta$. So this is just the left half our earlier $|x - a| < \delta$ requirement.

Of course we can make a similar definition for a one-sided limit from the right.

Definition 2.30. Suppose a is a real number, and f is a function defined on some open interval $(a, a+h)$ We say the *limit* of $f(x)$ as x approaches a *from the right* is L, and write

$$\lim_{x\to a^+} f(x) = L,$$

if for every real number $\epsilon > 0$ there is a real number $\delta > 0$ such that whenever $a < x < a+\delta$ then $|f(x) - L| < \epsilon$.

Remark 2.31. Notice that $|x - a| < \delta$ is the same as saying $-\delta < x - a < \delta$, which is the same as $a - \delta < x < a + \delta$. This makes our two-sided limit definition look a lot more like the one-sided definition.

Example 2.32. Let's compute $\lim_{x \to 0^-} H(x)$. It looks like this limit should be 0, since $H(x) = 0$ whenever $x < 0$.

So let $\epsilon > 0$ and $\delta = \underline{1}$. Then if $0 - 1 < x < 0$, then $|H(x) - 0| = |0 - 0| = 0 < \epsilon$.

Notice that $\lim_{x \to 0^-} H(x) \neq H(0)$. We say that $H(x)$ is "not continuous at 0", a concept we will discuss more in section 4.

Example 2.33. Now let's compute $\lim_{x \to 0^-} H(x)$. It looks like this limit should be 1, since $H(x) = 1$ whenever $x > 0$.

So let $\epsilon > 0$ and $\delta = \underline{1}$. Then if $0 < x < 0 + 1$, then $|H(x) - 1| = |1 - 1| = 0 < \epsilon$.

Example 2.34. Let

$$f(x) = \begin{cases} x^2 + 3 & x < -1 \\ 3x^2 & x \geq -1 \end{cases}$$

Find $\lim_{x \to -1^+} f(x)$.

From the right this looks like $3x^2$ so we guess 3.

Let $\epsilon > 0$ and set $\delta \leq \underline{1, \epsilon/9}$. Then if $-1 < x < -1 + \delta$, we compute

$$\begin{aligned} |f(x) - 3| &= |3x^2 - 3| && \text{since } x > -1 \\ &= 3|x^2 - 1| = 3|x - 1| \cdot |x + 1| < 3\delta|x - 1| \\ &= 3\delta|x + 1 + (-2)| \leq 3\delta(|x + 1| + |2|) \leq 3\delta(3) = 9\epsilon/9 = \epsilon. \end{aligned}$$