Math 114 Fall 2017
Calculus I HW 2 Solutions
Due Wednesday, September 13

1. Find, with proof, $\lim\limits_{x \to 3} 4x$.

   Solution: We guess $4 \cdot 3 = 12$.

   Let $\epsilon > 0$ and let $\delta = \underline{\epsilon/4}$. Then if $|x-3| < \delta$, we have

   $$|4x - 12| = |4(x-3)| = 4|x-3| < 4\delta = 4\epsilon/4 = \epsilon.$$

2. Find, with proof, $\lim\limits_{x \to 2} (x+1)^2$.

   Solution: We guess $(2+1)^2 = 9$.

   Let $\epsilon > 0$ and let $\delta < \underline{\epsilon/7, 1}$. Then if $|x-2| < \delta$ we have

   $$\begin{aligned} |(x+1)^2 - 9| &= |x^2 + 2x + 1 - 9| = |x^2 + 2x - 8| = |(x^2 - 4) + 2(x-2)| \\ &\leq |x-2| \cdot |x+2| + 2|x-2| \leq |x+2|\delta + 2\delta \\ &= \delta(2 + |x - 2 + 4|) \leq \delta(2 + |x-2| + 4) \leq \delta(7) < \epsilon. \end{aligned}$$

   Alternate Solutions: We can compute $|(x+1)^2 - 9| < \delta(6+\delta)$ and then solve the quadratic equation $\delta^2 + 6\delta - \epsilon = 0$ for δ, giving us $\delta = \frac{-6 \pm \sqrt{36 + 4\epsilon}}{2}$. Thus if $x < \delta = \frac{-6 + \sqrt{36+4\epsilon}}{2} = -3 + \sqrt{9 + \epsilon}$ then $|f(x) - 9| < \epsilon$.

   Alternatively again, we can observe

   $$|(x+1)^2 - 9| = |x^2 + 2x - 8| = |x+4| \cdot |x-2| < |x+4|\delta \leq \delta(|x-2| + 6)$$

   which then follows through into either of the previous solutions.

3. Find, with proof, $\lim\limits_{x \to 1} x^2$.

   Solution: We guess $1^2 = 1$.

   Let $\epsilon > 0$ and set $\delta < \underline{\epsilon/3, 1}$. Then if $|x-1| < \delta$ we compute

   $$\begin{aligned} |x^2 - 1| &= |x-1| \cdot |x+1| = |x-1| \cdot |x - 1 + 2| \leq |x-1|(|x-1| + 2) \\ &< \delta(1+2) < 3\epsilon/3 = \epsilon. \end{aligned}$$

Alternate Solutions: We can observe that $|x-1|(|x-1|+2) < \delta(\delta+2)$ and solve $\delta^2 + 2\delta - \epsilon = 0$ for δ, giving $\delta = \frac{-2\pm\sqrt{4+4\epsilon}}{2} = -1 \pm \sqrt{1+\epsilon}$. Then we observe that if $x < \delta = -1 + \sqrt{1+\epsilon}$ then $|x^2 - 1| < \epsilon$.

4. Find, with proof, $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$.

   Solution: We guess that we can cancel out an $x-3$, and thus get $3+3=6$.

   Let $\epsilon > 0$ and set $\delta = \underline{\epsilon}$. Then if $0 < |x+3| < \delta$, we compute

$$\left|\frac{x^2-9}{x-3} - 6\right| = \left|\frac{(x-3)(x+3)}{x-3} - 6\right| = |(x+3)-6|$$
$$= |x-3| < \delta = \epsilon.$$

5. $\star$ Find, with proof, $\lim_{x\to 2} \dfrac{1}{x-1}$.

   Solution: We guess $1/(2-1) = 1$.

   Let $\epsilon > 0$ and set $\delta < \underline{\epsilon/2, 1/2}$. Then if $|x-2| < \delta$, we compute

$$\left|\frac{1}{x-1} - 1\right| = \left|\frac{1-1(x-1)}{x-1}\right| = \left|\frac{2-x}{x-1}\right| = \frac{|x-2|}{|x-1|}$$

   But we know that

$$|x-1| = |x-2+1| = |1-(2-x)| \geq 1 - |x-2| \geq 1-\delta \geq 1/2$$

   and thus

$$\frac{|x-2|}{|x-1|} \leq \frac{\delta}{1-\delta} < \frac{\epsilon/2}{1/2} = \epsilon.$$

6. $(\star)$ Find (with proof) $\lim_{x\to 5} \frac{1}{x-4}$.

   Solution: Let $\epsilon > 0$ and let $\delta \leq \underline{1/2, \epsilon/2}$. Then if $|x-5| < \delta$, we compute

$$\left|\frac{1}{x-4} - 1\right| = \left|\frac{1-(x-4)}{x-4}\right|$$
$$= \frac{|5-x|}{|x-4|} < \frac{\delta}{|x-4|}.$$

   Since the denominator is positive, we need to make the denominator big, and so use the reverse triangle inequality. Then we compute

$$|x-4| = |(x-5)+1| = |1-(5-x)| \geq 1 - |5-x| > 1-\delta \geq 1/2$$

   after we set $\delta \leq 1/2$. Thus

$$\left|\frac{1}{x-4} - 1\right| < \frac{\delta}{|x-4|} < \frac{\delta}{1/2} = 2\delta < \epsilon.$$