

Math 114 Fall 2017
Calculus I HW 3 Solutions
Due Wednesday, September 20

1. Let

$$f(x) = \begin{cases} 1 & x < 2 \\ 2 & x = 2 \\ 3 & x > 2 \end{cases}$$

What is $f(2)$? Prove that $\lim_{x \rightarrow 2} f(x)$ does not exist.

Solution: $f(2) = 2$ by definition of the function.

Suppose $\lim_{x \rightarrow 2} f(x) = L$. Set $\epsilon = 1$; we can choose $\delta > 0$ so that when $0 < |x - 2| < \delta$ then $|f(x) - L| < \epsilon = 1$.

Let $x_1 = 2 + \delta/2$, so that $|x_1 - 2| = \delta/2 < \delta$; then $f(x_1) = 3$, and thus $1 > |f(x_1) - L| = |3 - L|$.

Let $x_2 = 2 - \delta/2$, so that $|x_2 - 2| = \delta/2 < \delta$; then $f(x_2) = 1$ and $1 > |f(x_2) - L| = |1 - L| = |L - 1|$. (We flip the sign inside the absolute value to make the next step easier, when we would like the L s to cancel).

Adding these two inequalities and applying the triangle inequality gives us

$$1 + 1 > |L - 1| + |3 - L| \geq |(L - 1) + (3 - L)| = 2,$$

but this is a contradiction, so the limit must not exist.

2. Let

$$j(x) = \begin{cases} 3x - 1 & x < 0 \\ 2x + 1 & x \geq 0 \end{cases}$$

Show that $\lim_{x \rightarrow 3} j(x) = 7$.

Solution: Let $\epsilon > 0$ and fix $\delta \leq \underline{3, \epsilon/2}$. Then if $0 < |x - 3| < \delta$ we know that $x > 0$, and so we have

$$|j(x) - 7| = |2x + 1 - 7| = |2x - 6| = 2|x - 3| < 2\delta \leq \epsilon.$$

3. (★) For the same function j , show that $\lim_{x \rightarrow 0} j(x)$ does not exist.

Solution: Suppose $\lim_{x \rightarrow 0} j(x) = L$. Fix $\epsilon = \underline{1}$ and suppose $\delta > 0$. Pick $x_1 = \delta/2$ and $x_2 = -\delta/2$.

Then since $0 < |x_1 - 0| = \delta/2 < \delta$, we have

$$\epsilon > |j(x_1) - L| = |2\delta/2 + 1 - L| = |\delta + 1 - L|.$$

Similarly, since $0 < |x_1 - 0| = \delta/2 < \delta$, we have

$$\epsilon > |j(x_1) - L| = |-3\delta/2 - 1 - L| = |L + 1 + 3\delta/2|.$$

Adding these equations gives

$$\begin{aligned} 2\epsilon &> |\delta + 1 - L| + |L + 1 + 3\delta/2| \\ &\geq |\delta + 1 - L + L + 1 + 3\delta/2| = |5\delta/2 + 2| \\ &= 2 + 5\delta/2. \end{aligned}$$

Since $\epsilon = 1$ this gives us $2 > 2 + 5\delta/2$ and thus $0 > \delta$ which is a contradiction. So no such limit exists.

4. (★) Prove that $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist.

Solution: Suppose $\lim_{x \rightarrow 0} \frac{|x|}{x} = L$. Set $\epsilon = 1$; we can choose $\delta > 0$ so that when $0 < |x| < \delta$ then $\left| \frac{|x|}{x} - L \right| < \epsilon = 1$.

Let $x_1 = \delta/2 < \delta$; then

$$1 = \epsilon > \left| \frac{|x_1|}{x_1} - L \right| = \left| \frac{|\delta/2|}{\delta/2} - L \right| = |1 - L|.$$

Let $x_2 = -\delta/2$ so that $|x_2| = \delta/2 < \delta$; then

$$1 = \epsilon > \left| \frac{|x_2|}{x_2} - L \right| = \left| \frac{|-\delta/2|}{-\delta/2} - L \right| = |-1 - L| = |L + 1|.$$

Then adding the two inequalities and using the triangle inequality gives us

$$2 = 1 + 1 > |L + 1| + |1 - L| \geq |(L + 1) + (1 - L)| = 2,$$

but this is a contradiction, so the limit must not exist.

5. Compute from the definition $\lim_{x \rightarrow 2} \frac{1}{x-2}$.

Solution: Let $N > 0$ and set $\delta = \frac{1}{N}$. Then if $0 < |x - 2| < \delta$, then

$$\left| \frac{1}{x-2} \right| = \frac{1}{|x-2|} > \frac{1}{\delta} = \frac{1}{1/N} = N.$$

Thus $\lim_{x \rightarrow 2} \frac{1}{x-2} = \pm\infty$.

6. Compute from the definition $\lim_{x \rightarrow 5} \frac{x-2}{x-5}$.

Solution: Let $N > 0$ and set $\delta \leq \underline{2, 1/N}$. Then if $0 < |x - 5| < \delta$, we have

$$\begin{aligned} \left| \frac{x-2}{x-5} \right| &= \frac{|x-2|}{|x-5|} > \frac{|x-2|}{\delta} \\ &= \frac{|x-5+3|}{\delta} \geq \frac{3-|x-5|}{\delta} \\ &> \frac{3-\delta}{\delta} \geq \frac{1}{\delta} \geq \frac{1}{1/N} = N. \end{aligned}$$

Thus $\lim_{x \rightarrow 5} \frac{x-2}{x-5} = \pm\infty$.

7. Compute from the definition $\lim_{x \rightarrow -3} \frac{-1}{(x+3)^4}$.

Solution: Let $N > 0$ and set $\delta = \underline{1/\sqrt[4]{N}}$. Then if $|x + 3| < \delta$, we have

$$\begin{aligned} \frac{1}{(x+3)^4} &> \frac{1}{\delta^4} = \frac{1}{1/N} = N \\ \frac{-1}{(x+3)^4} &< \frac{-1}{\delta^4} = \frac{-1}{1/N} = -N. \end{aligned}$$

Thus $\lim_{x \rightarrow -3} (x+3)^4 = -\infty$.