Lab 11 Tuesday December 5

Quadratic Approximation

In this class we've spent a lot of time on *linear approximation*: we can approximate a function with its tangent line, which is the linear function most similar to our starting function. This simplifies a lot of things, but is only an approximation.

$$f(x) \approx f(a) + f'(a)(x-a). \tag{1}$$

How good this approximation is depends on two things. The first is the distance $|x-a|$; the approximation is better when your goal point x is close to your starting point a. There are other techniques (like Fourier series) that don't have this limitation, but we won't discuss them in this course.

The other is the speed at which the derivative changes. If the derivative is constant, your function is just a line and the "approximation" is perfect. But the faster the derivative changes, the faster the function deviates from the line.

Thus we might try to get a better approximation using the second derivative, which tells us how quickly the derivative is changing. So how can we do this?

We're looking for some function $g(x)$ so that

$$f(x) \approx f(a) + f'(a)(x-a) + g(a)(x-a)^2.$$

(We want the linear approximation to be the same as (1), and we want the third derivative to be zero, so the only thing that can change at all is the degree two term). Taking derivatives of both sides gives us

$$\begin{aligned} f'(x) &\approx f'(a) + 2g(a)(x-a) \\ f''(x) &\approx 2g(a). \end{aligned}$$

Thus we set $g(a) = f''(x)/2$, and we get the equation

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2. \tag{2}$$

This is the *parabola* that best approximates our function near a.

(We could extend this logic to get a degree three approximation, or a degree four, etc. This is called a "Taylor Series" and will be covered at the end of Calculus 2).

We can use quadratic approximation to refine our estimates of functions. We usually use it in formulas, rather than to estimate specific numbers, since Newton's method is much more effective at estimating specific numbers.

Example

Let's estimate 1.01^{25} using a quadratic approximation. We use the function $f(x) = (1+x)^{25}$, and center our approximation at $x = 0$. (Equivalently we could consider $g(x) = x^{25}$ and center our approximation at $x = 1$; the way I set it up is a bit more common).

We take $f'(x) = 25(1+x)^{24}$ so $f'(0) = 25$, and $f''(x) = 25 \cdot 24(1+x)^{23}$ so $f''(0) = 25 \cdot 24 = 600$. Then we have

$$
\begin{aligned}
f(x) &\approx 1 + 25(x-0) + \frac{600}{2}(x-0)^2 = 1 + 25x + 300x^2 \\
1.01^{25} = f(.01) &\approx 1 + 25 \cdot .01 + 300 \cdot .0001 = 1 + .25 + .03 = 1.2
\end{aligned}
$$

Since $1.01^{25} \approx 1.28243$ this is pretty good.

What if we move a bit farther? If we want to estimate 1.04^{25} we get

$$1.04^{25} = f(.04) \approx 1 + 25 \cdot .04 + 300 \cdot .0016 = 1 + 1 + .48 = 2.48$$

while $1.04^{25} \approx 2.66584$. We've lost fidelity because our move away is bigger. We've lost it quickly because $f(x)$ has a huge *third* derivative.

Exercises

For all these exercises, I encourage you to plot the graphs of the true function and the quadratic approximation you're using.

1. Use a quadratic approximation to estimate $\sqrt{5}$ based at four. How does this compare to our linear estimate?
2. Use a quadratic approximation to estimate $\sqrt[3]{28}$.
3. If $f(x) = (x+5)^{1/2}$, compute a quadratic approximation centered at $x = 1$. Use this to estimate $f(1.02)$.
4. Let $g(x) = x^4 - 3x^3 + 4x^2 + 4x - 2$. Compute the linear approximation at $x = -2$. Compare that to $g(x)$. Use this to estimate $g(-1.97)$.
5. Compute the quadratic approximations of $\sin(x)$ and $\cos(x)$ centered at zero. Estimate $\sin(.01)$ and $\cos(.01)$? How does this relate to the Small Angle Approximation?
6. Compute the quadratic approximation of e^x centered at zero. Estimate $e^{.1}$.
7. Compute the quadratic approximation of $\ln(1+x)$ centered at zero. Use this to estimate $\ln(1.1)$ and $\ln(2)$. How accurate do you expect these approximations to be? Check the true answers in Mathematica.
8. If $f(x) = e^{x+x^2}$, find a formula for the quadratic approximation near zero, and use that to estimate $f(-.1)$.
9. Compute the quadratic approximation of $(1+x)^\alpha$ centered at 0. Use this formula to estimate 2^{10}. Use it to estimate 1.1^{10}.

Bonus: Special Relativity

Many formulas in the theory of special relativity depend on a parameter

$$\gamma(v) = \frac{1}{\sqrt{1-(v/c)^2}}$$

where v is the velocity, and c is the speed of light.

(a) What is $\gamma(0)$?

(b) Compute formulas for the linear and quadratic approximations to $\gamma(v)$ centered at zero. These tell us what happens when v is small relative to the speed of light.

(c) You are probably familiar with the famous formula that $E = mc^2$. This formula is for "rest energy", and holds when $v = 0$. For a moving object, we can compute the kinetic energy at a given velocity with the formula

$$E(v) = mc^2\gamma(v).$$

What happens if we replace γ with the linear approximation? Does this look familiar?

What happens if we take a quadratic approximation instead?