Lab 8 Tuesday November 21

Exponential Growth

Discrete Compound Growth

Often we have something that grows at a percentage rate. Say a population of bacteria is growing at a rate of 50% per hour. If it starts at 1000, then after an hour there will be 1500. After two hours there will be, not 2000, but 2250. We see that after t hours the population will be $1000 \cdot (1.5)^t$. We can write this formula as

$$P(t) = P(0) \cdot (1 + r)^t. \tag{1}$$

Interest payments in particular have an interesting extra issue. In the example above the interest was *compounded* once per year–meaning that each year a payment was made, taking into account the amount of money after the previous year. But sometimes payments were compounded more often. Rather than making a 4% payment every year, borrowers would make a 1% payment each of four times per year. In this case we can generalize our formula: if your payment is compounded n times per year, then

$$P(t) = P(0) \cdot (1 + r/n)^{tn}. \tag{2}$$

Continuous compound growth

The mathematician Jacob Bernoulli in the late 1600s was interested in the idea of *continuously compounded interest*, which is the limit as the rate of compounding goes to infinity. This also makes more sense in many natural situations: most growth doesn't happen in distinct instantaneous spikes, but happens continuously over time. So we might expect this to show up in nature.

Mathematically, Bernoulli asked for the limit $\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n$. You might recognize this from the definition of e; in fact, we have

$$\lim_{n\to+\infty} (1 + 1/n)^n = e$$

and thus Bernoulli's limit was

$$\lim_{n\to+\infty} (1 + r/n)^n = \lim_{n/r\to+\infty} \left(1 + \frac{1}{n/r}\right)^{rn/r} = \left(\lim_{n/r\to+\infty} \left(1 + \frac{1}{n/r}\right)^{n/r}\right)^r = e^r.$$

Thus when something is epxeriencing continuous compound growth at a rate of r, we have

$$P(t) = P(0) \cdot e^{rt}. \tag{3}$$

We can find this same equation a totally different way, through differential equations. In these situations of compound growth, the derivative of the population is not a constant–as the population gets bigger, it grows faster. (An increase of 50% is bigger when you start with 1000 than when you start with 10). So the derivative $P'(t)$ is definitely not constant.

However, the "growth rate" as a proportion of the total population is a constant. We can express this as

$$P'(t) = rP(t) \tag{4}$$

for some rate r, which we call the *relative growth rate*.

But we can see that the function of equation (3) satisfies this differential equation! For

$$P'(t) = P(0) \cdot e^{rt} \cdot r = rP(t).$$

In Calculus 2 you will see that this solution is unique up to a constant factor (which simply represents the initial population). Thus equation (3) is the only one we could possibly have if the relative growth rate is a constant.

Thus whenever we know we have a constant relative growth rate, we know it must be described by equation (3). We then only need to find two points to determine $P(0)$ and r, in order to find a complete equation.

Exercises

1. **Compound Interest** If you invest \$100 at a rate of 4%, how much will you have after ten years if you compound annually? What if you compound quarterly? Four hundred times per year? Which would you prefer?

   What if you compound continuously? **Solution:**

$$\begin{aligned}\text{annually: } 100(1.04)^{10} &= 148.024\\ \text{quarterly: } 100(1.01)^{40} &= 148.886\\ 400\text{: } 100(1.0001)^{4000} &= 149.179\end{aligned}$$

   We see we get the most money if we compound the most often, at four hundred times per year.

   For continuous compounding we get

$$100e^{.04 \cdot 10} \approx 149.182$$

   This is marginally better than compounding four hundred times per year. But not by much!

2. **Population modelling**

   (a) In real life, we often want to find solutions to differential equations in order to model something we've observed. To do this we need an "initial condition"–some particular data point. If a population is growing at a monthly rate of 10% and has 100 members at the start of the process, what is the equation for $P(t)$?

   Solution:

$$P(t) = 100e^{(}.1t)$$

   (b) What if the population is growing at a monthly rate of 5% and has 116 members after three months (at $t = 3$)? What is the equation in this case?

   Solution: We have $P(t) = Ce^{.05t}$ and $P(3) = 116$. Thus $116 = Ce^{.05 \cdot 3}$ and thus $C = 116/e^{.15} \approx 100$.

   (c) What if the population is growing at a yearly rate of 7% and has 71 members after five years?

   Solution: We have $P(t) = Ce^{.07t}$ and $P(5) = 71$. Thus $71 = Ce^{.35}$ and $C = 71/e^{.35} \approx 50$.

3. **Human Global Population Modelling**

   We can use these tools to model the global growth of **human** population. If we assume population growth is exponential, we generally get pretty reasonable numbers.

   (a) The population of earth was 3 billion people in 1960, and 4 billion people in 1975. Write an equation giving the population of the earth as a function of time.

   Solution: It's easiest to take $t = 0$ to be the year 1960, and measure in billions. We have $P(t) = Ce^{rt}$, and we know $P(0) = 3$ and $P(15) = 4$. Thus $3 = Ce^0 = C$, and we have $4 = 3e^{15r}$. This gives us $\ln(4/3) = 15r$ adn thus $r = \ln(4/3)/15$, and therefore

$$P(t) = 3e^{\ln(4/3)t/15}.$$

If you want a decimal approximation, we can compute that $\ln(4/3)/15 \approx .019$.

(b) What do you estimate the population to be in 2020?

Solution: $P(60) = 3e^{\ln(4/3)\cdot 60/15} = 3e^{4\ln(4/3)} = 3(4/3)^4 = \frac{2^8}{27} \approx 9.48$. Thus we estimate about 9.48 billion people in 2020.

(c) In what year would you expect population to hit 12 billion?

Solution: We solve the equation $12 = 3e^{\ln(4/3)t/15}$ and thus $\ln(4) = \ln(4/3)t/15$, hence $t = 15\ln(4)/\ln(4/3) \approx 72$. Thus we expect the population to hit 12 billion in 2032.

(d) The actual population in 2020 is estimated to be 7.7 billion. What does this tell you about your model?

Solution: Our model is an overestimate. This suggests that the rate of population growth has been and is decreasing. (Indeed, we get much better results with a model called *logistic growth*, which takes into account the idea that growth rates might decline as populations get closer to carrying capacity).

Implict Functions and their Tangents

When using the `ContourPlot` command, note the double `==` signs.

1. In class, we showed that the tangent line to $x^3 + y^3 = 6xy$ at the point $(3, 3)$ is $y - 3 = 3 - x$. Verify this with the command
   `ContourPlot[{x^3 + y ^3 == 6 x*y, y-3 == 3 -x}, {x,-5,5},{y,-5,5}]`

2. (a) Use ContourPlot to plot the "cardioid" with equation:
   `x^2 + y^2 == ( 2x^2 + 2y^2 -x)^2.` (x and y domains from -1 to 1).

   (b) Compute the derivative at the point $(0, 1/2)$ by hand.

   (c) Check your computation by running the commands
   `D[x^2 + y[x]^2 == (2x^2 + 2y[x]^2 - x)^2,x]` and
   `D[x^2 + y[x]^2 == (2x^2 + 2y[x]^2 - x)^2,x] /. y[x] -> 1/2 /.x -> 0`

Note some important details here. Mathematica can't figure out that `y` is a function of `x` instead of a constant unless we tell it, so we write `y[x]` instead of `y`. We can have Mathematica automatically substitute for us, but it matters that we do `/.y[x]->1/2` before `/.x->0`. Why? Try it the other way and see what happens.

(d) Plot the tangent line to the cardioid at that point in Mathematica.

(e) What do you expect to happen if you try to find the tangent line at $(0,0)$? Are you right? What does Mathematica say?

(f) Looking at the graph, what do you think is the tangent line at the point $(1,0)$? Can you get this from your derivative formula? Try computing the (implicit) derivative with respect to y instead of x. What happens?

3. (a) Plot the "devil's curve" `y^2(y^2 -4) == x^2 (x^2 - 5)`
   (b) Compute the derivative at $(0,-2)$.
   (c) Plot the devil's curve and its tangent line simultaneously.
   (d) Run the command `ContourPlot[y^2(y^2-4) - x^2(x^2 -5),{x,-5,5},{y,-5,5}]` What happens? Why?

4. (a) Plot `(x^2 + y^2 -1)^3 - x^2 * y^3 == 0`
   (b) Check that $(1,1)$ is a solution to this equation, and compute the derivative at $(1,1)$.
   (c) Plot the tangent line.
   (d) Now try plotting without the equals sign, as in (3).

5. (a) Plot `Sin[x^2 + y^2] == Cos[x * y]` from -5 to 5.
   (b) As before, replace the `==` with a `-` sign.

Just Because They're Pretty

1. Some other functions to try:
   - `Sin[Sin[x] + Cos[y]] == Cos[Sin[x * y] + Cos[y]]`
   - `Abs[Sin[x^2 - y^2]] == Sin[x + y] + Cos[x * y]`
   - `Csc[1-x^2] * Cot[2-y^2] == x * y`
   - `Abs[Sin[x^2 + 2 * x * y]] == Sin[x - 2 y]`
   - `(x^2 + y^2 - 3) Sqrt[x^2 + y^2] + .75 + Sin[4 Sqrt[x^2  + y^2]] Cos[84 ArcTan[y/x]] - Cos[6 ArcTan[y/x]] ==  0`

2. Try replacing the `==` signs with `-` signs.

3. Look at the examples on the Wolfram Alpha page `https://www.wolframalpha.com/examples/PopularCurves.html`