

**Problem 1.** (a) Use the definition of limit to prove that  $\lim_{x \rightarrow 2} \frac{1}{x+3} = \frac{1}{5}$ .

**Solution:** Let  $\epsilon > 0$  and set  $\delta \leq \underline{20\epsilon, 1}$ . Then if  $|x - 2| < \delta$  we have

$$\begin{aligned} \left| \frac{1}{x+3} - \frac{1}{5} \right| &= \left| \frac{2-x}{5(x+3)} \right| = \frac{|x-2|}{5|x-2+5|} \\ &= \frac{|x-2|}{5|5-(2-x)|} \leq \frac{|x-2|}{5(5-|x-2|)} \\ &< \frac{\delta}{5(5-\delta)} \leq \frac{20\epsilon}{20} = \epsilon. \end{aligned}$$

(b) Use the definition of limit to prove that  $\lim_{x \rightarrow 1} \frac{1}{(x-1)^2} = +\infty$ .

**Solution:** Let  $N > 0$  and set  $\delta = \frac{1}{\sqrt{N}}$ . Then if  $|x - 1| < \delta$  we have

$$\frac{1}{(x-1)^2} = \frac{1}{|x-1|^2} > \frac{1}{\delta^2} = \frac{1}{1/N} = N.$$

**Problem 2.** (a) Use the Squeeze Theorem to show that  $\lim_{x \rightarrow 5} (x-5) \sin\left(\frac{x^2+1}{x-5}\right) = 0$ .

**Solution:** We have

$$\begin{aligned} -1 &\leq \sin\left(\frac{x^2+1}{x-5}\right) \leq 1 \\ -|x-5| &\leq (x-5) \sin\left(\frac{x^2+1}{x-5}\right) \leq |x-5|. \end{aligned}$$

We see that  $\lim_{x \rightarrow 5} -|x-5| = \lim_{x \rightarrow 5} |x-5| = 0$ , so by the squeeze theorem we know that

$$\lim_{x \rightarrow 5} (x-5) \sin\left(\frac{x^2+1}{x-5}\right) = 0.$$

(b) Compute  $\lim_{x \rightarrow 25} \frac{\sqrt{x} - 5}{x - 25}$

**Solution:**

$$\lim_{x \rightarrow 25} \frac{\sqrt{x} - 5}{x - 25} = \lim_{x \rightarrow 25} \frac{x - 25}{(x - 25)(\sqrt{x} + 5)} = \lim_{x \rightarrow 25} \frac{1}{\sqrt{x} + 5} = \frac{1}{10}.$$

(c) Compute  $\lim_{x \rightarrow 0} \frac{\sin(x^2)}{\sin^2(x)}$

**Solution:** We use the small angle approximation. We rewrite this as

$$\lim_{x \rightarrow 0} \frac{\sin(x^2)}{\sin^2(x)} = \lim_{x \rightarrow 0} \frac{\sin(x^2)}{x^2} \frac{x}{\sin(x)} \frac{x}{\sin(x)} = 1.$$

**Problem 3.** (a) **Directly from the definition**, compute  $f'(1)$  where  $f(x) = \sqrt{x+3}$ .

**Solution:**

$$\begin{aligned} f'(1) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(4+h) - 4}{h(\sqrt{4+h} + \sqrt{4})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{4+h} + 2} = \frac{1}{4}. \end{aligned}$$

(b) Compute  $g'(x)$  where  $g(x) = \ln \left| \frac{e^{\arctan(x^2)} - 5}{\sqrt[4]{x^2 + 1}} \right|$ .

**Solution:**

$$g'(x) = \frac{1}{\frac{e^{\arctan(x^2)} - 5}{\sqrt[4]{x^2 + 1}}} \cdot \frac{(e^{\arctan(x^2)} \frac{2x}{1+x^4}) \sqrt[4]{x^2 + 1} - \frac{1}{4}(x^2 + 1)^{-3/4} 2x (e^{\arctan(x^2)} - 5)}{\sqrt[2]{x^2 + 1}}$$

(c) Find a tangent line to the function  $f(x) = \frac{e^x}{x}$  at the point given by  $x = 2$ .

**Solution:**

$$f'(x) = \frac{e^x \cdot x - e^x}{x^2},$$

so  $f'(2) = \frac{2e^2 - e^2}{4} = \frac{1}{4}e^2$ . Thus the tangent line has equation

$$y = \frac{1}{4}e^2(x - 2) + \frac{1}{2}e^2.$$

**Problem 4.** (a) **Directly from the definition**, compute  $f'(x)$  where  $f(x) = \frac{1}{x-7}$ .

**Solution:**

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h-7} - \frac{1}{x-7}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x-7) - (x+h-7)}{(x-7)(x+h-7)h} = \lim_{h \rightarrow 0} \frac{-1}{(x-7)(x+h-7)} = \frac{-1}{(x-7)^2}. \end{aligned}$$

(b) Write a tangent line to the curve  $y^2 = x^{x \cos(x)}$  at the point  $(\pi/2, -1)$ .

**Solution:** Implicit differentiation gives us

$$\begin{aligned} 2 \ln(y) &= x \cos(x) \ln(x) \\ \frac{2y'}{y} &= \cos(x) \ln(x) - x \sin(x) \ln(x) + \cos(x) \\ y' &= \frac{1}{2} (\cos(x) \ln(x) - x \sin(x) \ln(x) + \cos(x)) y. \end{aligned}$$

When  $x = \pi/2, y = -1$ , this gives us

$$\begin{aligned} y' &= \frac{1}{2} (0 \ln(\pi/2) - \pi/2 \cdot 1 \cdot \ln(\pi/2) + 0) (-1) = \frac{1}{2} (\pi/2 \ln(\pi/2)) \\ &= \frac{\pi(\ln(\pi) - \ln(2))}{4} \end{aligned}$$

and thus the tangent line has equation

$$y = \frac{\pi(\ln(\pi) - \ln(2))}{4} (x - \pi/2) - 1.$$

(c) Find  $y'$  if  $e^y + \ln(y) = x^2 + 1$ .

**Solution:**

$$\begin{aligned} e^y \cdot y' + \frac{y'}{y} &= 2x \\ y'(e^y + \frac{1}{y}) &= 2x \\ y' &= \frac{2x}{e^y + \frac{1}{y}}. \end{aligned}$$

**Problem 5.** (a) A cone with height  $h$  and base radius  $r$  has volume  $\frac{1}{3}\pi r^2 h$ . Suppose we have an inverted conical water tank, of height 4m and radius 6m. Water is leaking out of a small hole at the bottom of the tank. If the current water level is 2m and the water level is dropping at  $\frac{1}{9\pi}$  meters per minute, what volume of water leaks out every minute?

**Solution:** We have  $V = \frac{1}{3}\pi r^2 h$  and  $r = 3h/2$ , and thus

$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{3h}{2}\right)^2 h = \frac{3}{4}\pi h^3 \\ V' &= \frac{9}{4}\pi h^2 h' \\ V' &= \frac{9}{4}\pi(2)^2 \frac{-1}{9\pi} = 1 \end{aligned}$$

So one cubic meter of water is leaking out every minute.

(b) Use two iterations of Newton's method, starting at 0, to estimate the root of  $e^x - 3x$ .

**Solution:** Set  $f(x) = e^x - 3x$ , and  $x_1 = 0$ . We have  $f'(x) = e^x - 3$ .

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{1 - 0}{1 - 3} = \frac{1}{2} \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{1}{2} - \frac{\sqrt{e} - \frac{3}{2}}{\sqrt{e} - 3} = \frac{\sqrt{e} - 3}{2(\sqrt{e} - 3)} - \frac{2\sqrt{e} - 3}{2(\sqrt{e} - 3)} = \frac{\sqrt{e}}{6 - 2\sqrt{e}}. \end{aligned}$$

(c) Let  $g(x) = \sqrt[5]{x^9 + x^7 + x + 1}$ . Find  $(g^{-1})'(1)$ .

**Solution:** We see that  $g(0) = 1$ , so  $g^{-1}(1) = 0$ . Then by the Inverse Function Theorem we have

$$\begin{aligned} (g^{-1})'(1) &= \frac{1}{g'(g^{-1}(1))} = \frac{1}{g'(0)} \\ g'(x) &= \frac{1}{5}(x^9 + x^7 + x + 1)^{-4/5}(9x^8 + 7x^6 + 1) \\ g'(0) &= \frac{1}{5}(1)(1) = \frac{1}{5} \\ (g^{-1})'(1) &= 5. \end{aligned}$$

**Problem 6.** (a) If  $f(x) = \sqrt{x} + \tan(\pi x)$ , use a linear approximation centered at 4 to estimate  $f(4.1)$ .

**Solution:** We have  $f'(x) = \frac{1}{2\sqrt{x}} + \pi \sec^2(\pi x)$  so  $f'(4) = \frac{1}{4} + \pi$ . Then

$$\begin{aligned} f(x) &\approx f(4) + f'(4)(x - 4) = 2 + 0 + (\pi + 1/4)(x - 4) \\ f(4.1) &\approx 2 + \frac{\pi}{10} + \frac{1}{40} = \frac{81}{40} + \frac{\pi}{10}. \end{aligned}$$

(b) If  $g(x) = \cos(x)$ , use a quadratic approximation centered at 0 to estimate  $g(.1)$ .

**Solution:** We have  $g'(x) = -\sin(x)$  and  $g''(x) = -\cos(x)$ . So  $g'(0) = 0$  and  $g''(0) = -1$ , and then we have

$$\begin{aligned} g(x) &\approx g(0) + g'(0)(x - 0) + \frac{g''(0)}{2}(x - 0)^2 = 1 + 0x - \frac{1}{2}x^2 = 1 - x^2/2 \\ g(.1) &\approx 1 - .1^2/2 = .995. \end{aligned}$$

(c) Let  $g'(x) = g(x) + 3x$ , and  $g(2) = 4$ . Use two steps of Euler's method to estimate  $g(4)$ . Is this an overestimate or an underestimate?

**Solution:**

$$\begin{aligned} g(3) &\approx g'(2)(3 - 2) + g(2) = 10(1) + 4 = 14 \\ g(4) &\approx g'(3)(4 - 3) + g(3) = 23(1) + 14 = 37. \end{aligned}$$

This is a wild underestimate because the derivative is increasing so rapidly.

**Problem 7.** (a) Find the absolute extrema of  $f(x) = 3x^4 - 20x^3 + 24x^2 + 7$  on  $[0, 5]$ .

**Solution:**  $f$  is a continuous function on a closed interval, so it must have an absolute maximum and an absolute minimum.  $f'(x) = 12x^3 - 60x^2 + 48x = 12x(x^2 - 5x + 4) = 12x(x - 4)(x - 1)$  is defined everywhere and has roots at 0, 1, 4. The endpoints are 0, 5, so we need to evaluate  $f$  at 0, 1, 4, 5.

$$f(0) = 7$$

$$f(1) = 14$$

$$f(4) = 3(4^4) - 5(4^4) + \frac{3}{2}(4^4) + 7 = \frac{-1}{2}4^4 + 7 = 7 - 128 = -121$$

$$f(5) = 3 \cdot 5^4 - 4 \cdot 5^4 + 5^4 - 5^2 + 7 = 7 - 25 = -18.$$

So the absolute maximum is 14 at 1, and the absolute minimum is  $-121$  at 4.

(b) Find all the critical points of  $g(x) = \ln(x^3 + 9x^2 + 27x)$ . **Solution:** We have

$$g'(x) = \frac{3x^2 + 18x + 27}{x^3 + 9x^2 + 27x} = 3 \frac{(x + 3)^2}{x(x^2 + 9x + 27)}.$$

$g'(x)$  is undefined when  $x = 0$ , but  $g$  is also defined there, so this isn't really a critical point. (I'd give credit either way).  $g'(x) = 0$  when  $(x + 3)^2 = 0$  when  $x = -3$ . So the sole true critical point is  $-3$ .

(The original version of this problem had  $g(x) = \ln(x^3 + 3x^2 + 9)$ , which has derivative

$$g'(x) = 3 \frac{x^2 + 2x + 3}{x(x^2 + 9x + 27)}.$$

The numerator has no roots and the denominator has no roots except 0, so the function had no critical points at all).

(c) Classify the relative extrema of  $h(x) = \sqrt[3]{x}(x + 4)$

**Solution:** We have

$$h'(x) = \sqrt[3]{x} + \frac{1}{3}x^{-2/3}(x + 4) = \frac{x}{\sqrt[3]{x^2}} + \frac{x + 4}{3\sqrt[3]{x^2}} = \frac{4x + 4}{3\sqrt[3]{x^2}}$$

so  $h'(x)$  is undefined at  $x = 0$  and  $h'(x) = 0$  at  $x = -1$ . Thus the critical points are 0,  $-1$ . Those are the possible relative extrema.

We can classify these points in two ways. We can use the first derivative test or the second derivative test. In these solutions I'll do both.

For the second derivative test we compute:

$$h''(x) = \frac{4(3\sqrt[3]{x^2}) - \frac{4}{3}(x + 1)\frac{-2}{3}x^{-5/3}}{9\sqrt[3]{x^4}} = \frac{12\sqrt[3]{x^2} + \frac{8}{3}(x + 1)x^{-5/3}}{9\sqrt[3]{x^4}}$$

$$h''(-1) = \frac{12 + 0}{9} = \frac{4}{3} > 0$$

$$h''(0) = \frac{0 + 0}{0} \text{ is undefined}$$

So we see that  $h$  has a local minimum at  $-1$  since  $h''(-1) > 0$ , but this tells us nothing about the critical point at 0; the second derivative test is inconclusive there. So we're forced to use the first derivative test.

For the first derivative test we make a chart:

	$4x + 4$	$\frac{1}{3\sqrt[3]{x^2}}$	$h'(x)$
$x < -1$	-	+	-
$-1 < x < 0$	+	+	+
$0 < x$	+	+	+

so  $h$  has a relative minimum at  $-1$  and neither a maximum nor a minimum at 0.

(The first derivative test was definitely the easier path here).

**Problem 8.** (a) Find all the critical points of  $g(x) = \frac{x^2 - 8}{x + 3}$

**Solution:** The function is undefined at  $x = -3$ .

$g'(x) = \frac{2x(x+3) - 1(x^2-8)}{(x+3)^2} = \frac{x^2+6x+8}{(x+3)^2}$ . The denominator is zero when  $x = -3$ , and thus the derivative is undefined there, but so is the function. The numerator is  $(x+2)(x+4)$  and thus has roots when  $x = -2, -4$ . So the critical points of the function are  $-2$  and  $-4$ .

(b) If  $-1 \leq f'(x) \leq 3$  and  $f(0) = 0$ , what can you say about  $f(4)$ ? Assume  $f$  is continuous and differentiable.

**Solution:** By the Mean Value Theorem, there is some  $c$  such that  $f'(c) = \frac{f(4)-f(0)}{4-0}$ . Since  $-1 \leq f'(c) \leq 3$ , we have

$$-1 \leq \frac{f(4) - f(0)}{4} \leq 3$$

$$-4 \leq f(4) - 0 \leq 12$$

$$-4 \leq f(4) \leq 12$$

so  $f(4)$  is between  $-4$  and  $12$ .

(c) Prove that  $x^2 - (e^2 + 1)\ln(x)$  has at least two real roots.

**Solution:** Let  $g(x) = x^2 - (e^2 + 1)\ln(x)$ . Then  $g$  is continuous and differentiable for all real numbers greater than 0. We see that  $g(1) = 1 > 0$ ,  $g(e) = -1 < 0$ , and  $g(e^2) = e^4 - 2e^2 - 2 > 0$ . So by the intermediate value theorem,  $g$  has a root between 1 and  $e$ , and another between  $e$  and  $e^2$ .

**Problem 9.** Let  $j(x) = x^4 - 14x^2 + 24x + 6$ . We can compute  $j'(x) = 4(x+3)(x-1)(x-2)$  and  $j''(x) = 4(3x^2 - 7)$ . Sketch a graph of  $j$ .

**Solution:** The domain of  $j$  is all reals. I'm not going to worry about finding roots now, and there are no obvious symmetries. It's a polynomial of even degree, so it's easy to see that  $\lim_{x \rightarrow \pm\infty} j(x) = +\infty$ .

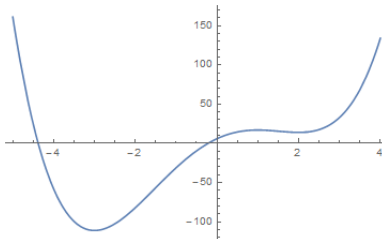
The function  $j$  is defined everywhere and is zero at three points. Thus  $j$  has three critical points, at  $-3, 1, 2$ . We compute  $j$  at these critical points:  $j(-3) = 81 - 126 - 72 + 6 = -111$ ,  $j(1) = 1 - 14 + 24 + 6 = 17$ ,  $j(2) = 14$ .

We can make a chart to determine when  $j$  increases or decreases:

	$(x+3)$	$(x-1)$	$(x-2)$	$j'(x)$
$x < -3$	-	-	-	-
$-3 < x < 1$	+	-	-	+
$1 < x < 2$	+	+	-	-
$2 < x$	+	+	+	+

So  $j$  is increasing between  $-3$  and  $1$  and when bigger than  $2$ , and  $j$  is decreasing when smaller than  $-3$  or between  $1$  and  $2$ . This implies that  $j$  has a relative minimum (of  $-111$ ) at  $-3$ , a relative maximum (of  $17$ ) at  $1$ , and a relative minimum of  $14$  at  $2$ .

$j''(x) = 4(3x^2 - 7)$  is defined everywhere, and is zero when  $x^2 = 7/3$ , when  $x = \pm\sqrt{7/3}$ .  $j''(x)$  is positive when  $|x| > \sqrt{7/3}$  and negative when  $|x| < \sqrt{7/3}$ .



Graph of  $j(x)$

**Problem 10.** Let  $g(x) = \arctan(x^2 + x)$ . We can compute that  $g'(x) = \frac{2x+1}{1+(x^2+x)^2}$  and

$$g''(x) = \frac{-6x^4 - 12x^3 - 8x^2 - 2x + 2}{(1 + (x^2 + x)^2)^2}.$$

Sketch a graph of  $g$ .

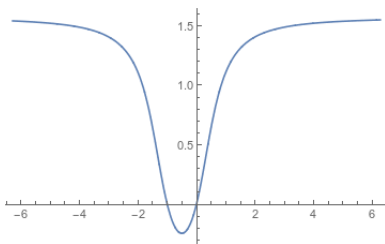
**Solution:**

$g$  has domain all reals, and no really useful symmetries.  $g(x) = 0$  when  $x^2 + x = 0$  when  $x = 0$  or  $x = -1$ .  $\lim_{x \rightarrow \pm\infty} x^2 + x = +\infty$ , so  $\lim_{x \rightarrow \pm\infty} g(x) = \pi/2$ .

$g'(x) = \frac{1}{1+(x^2+x)^2}(2x+1) = \frac{2x+1}{1+(x^2+x)^2}$ . The denominator has no roots, so the function is defined everywhere. The numerator is zero when  $x = -1/2$ , so the only critical point of  $g$  is  $x = -1/2$ .  $g'(-1) = -1$  and  $g'(0) = 1$ , so  $g$  is decreasing for  $x < -1/2$  and increasing for  $x > -1/2$ . Thus  $g$  has a minimum at  $-1/2$ .

$$g''(x) = \frac{-6x^4 - 12x^3 - 8x^2 - 2x + 2}{(1 + (x^2 + x)^2)^2}$$

The denominator is positive everywhere. It's clear that  $g''(0) = 2 > 0$ , but the numerator is negative if  $x = 1$  or if  $x = -2$ , so there is an inflection point between 0 and 1 and another between 0 and  $-2$ . These are the only inflection points, so the function is concave up near 0 and concave down in the tails.



The graph of  $g$  from  $-2\pi$  to  $2\pi$