Math 114 Practice Test 1 Solutions

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Problem 1.

(a) Directly from the definition of a limit, compute with proof $\lim_{x \to -2} \frac{x}{x+4}$ **Solution:** We guess -1. Let $\epsilon > 0$ and let $\delta \leq \underline{1, \epsilon/2}$. Then if $0 < |x+2| < \delta$, we compute

$$\left| \frac{x}{x+4} + 1 \right| = \left| \frac{2x+4}{x+4} \right| = \frac{2|x+2|}{|x+4|}$$

And we compute

$$|x+4| = |(x+2)+2| \geq 2 - |x+2| > 2 - \delta \geq 1$$

by the reverse triangle inequality. So

$$\left| \frac{x}{x+4} + 1 \right| = \frac{2|x+2|}{|x+4|} < \frac{2\delta}{1} < \epsilon.$$

(b) Directly from the definition, compute with proof $\lim_{x \to 3} \frac{2x^2 - 10x + 12}{x-3}$.

Solution: Let $\epsilon > 0$ and set $\delta \leq \underline{\epsilon/2}$. Then if $0 < |x-3| < \delta$ then

$$\begin{aligned} \left| \frac{2x^2 - 10x + 12}{x-3} - 2 \right| &= \left| \frac{2(x-3)(x-2)}{x-3} - 2 \right| \\ &= |2(x-2) - 2| = 2|x-3| < 2\delta \leq \epsilon. \end{aligned}$$

Problem 2.

Let

$$f(x) = \begin{cases} 5 & x < -1 \\ 2 & x > -1 \end{cases}$$

(a) Directly from the definition, compute with proof $\lim_{x \to 1} f(x)$.

Solution: Let $\epsilon > 0$ and set $\delta = \underline{2}$. Then if $0 < |x-1| < \delta$, we see that $x > -1$ and thus we have

$$|f(x) - 2| = |2 - 2| = 0 < \epsilon.$$

(b) Directly from the definition of a limit, prove that $\lim_{x \to -1} f(x)$ does not exist.

Solution: Set $\epsilon = 1$ and suppose $\delta > 0$. Suppose $\lim_{x \to -1} f(x) = L$. Then set $x_1 = -1 + \delta/2, x_2 = -1 - \delta/2$, and we have

$$\begin{aligned} \epsilon &> |f(x_1) - L| = |f(-1 + \delta/2) - L| = |2 - L| \\ \epsilon &> |f(x_2) - L| = |f(-1 - \delta/2) - L| = |5 - L| \\ 2\epsilon &> |L - 2| + |5 - L| \geq |L - 2 + 5 - L| = |3| = 3 \end{aligned}$$

Thus we have $3 < 2\epsilon = 2$ which is impossible.

Problem 3.

Let

$$g(x) = \begin{cases} x-3 & x<3 \\ 2x+1 & x>3 \end{cases}$$

(a) Directly from the definition, compute with proof $\lim_{x\to 0} g(x)$.

Solution: Let $\epsilon > 0$ and set $\delta \leq \underline{3, \epsilon}$. Then if $0 < |x-0| < \delta$ we have $x < 3$ and thus we compute

$$|g(x)+3| = |x-3+3| = |x| < \delta \leq \epsilon.$$

Thus the limit is -3.

(b) Directly from the definition of a limit, prove that $\lim_{x\to 3} g(x)$ does not exist.

Solution:

Suppose $\lim_{x\to 3} g(x) = L$. Set $\epsilon = \underline{3}$ and let $\delta > 0$. Let $x_1 = 3 - \delta/2$ and $x_2 = 3 + \delta/2$. Then we have

$$\begin{aligned} \epsilon &> |g(x_1) - L| = |3-\delta/2-3-L| = |-\delta/2-L| = |L+\delta/2| \\ \epsilon &> |g(x_2) - L| = |2(3+\delta/2)+1-L| = |7+\delta-L| = |L-7-\delta| \\ 2\epsilon &> |-\delta/2-L| + |L-7-\delta| \geq |-7-3\delta/2| = 7+3\delta/2 > 7. \end{aligned}$$

Since $\epsilon = 3$ this gives us $6 > 7$, which impossible. So no such limit exists.

Problem 4. (a) Directly from the definition, prove that $\lim_{x\to-\infty} \frac{x}{x-3} = 1$.

Solution:

Let $\epsilon > 0$ and set $M \geq \underline{3/\epsilon - 3}$. Then if $x < -M$, we know have

$$\left|\frac{x}{x-3} - 1\right| = \left|\frac{x-(x-3)}{x-3}\right| = \frac{3}{|x-3|} = \frac{3}{3-x} < \frac{3}{3+M} \leq \epsilon.$$

Alternatively, we could argue that

$$\left|\frac{x}{x-3} - 1\right| = \left|\frac{x-(x-3)}{x-3}\right| = \frac{3}{|x-3|}.$$

Then we make a sidebar: $x < -M$ so $x-3 < -M-3$ so $|x-3| > M+3$, and $\frac{3}{|x-3|} < \frac{3}{M+3}$. Thus we have

$$\frac{3}{|x-3|} < \frac{3}{M+3} \leq \epsilon$$

in the same manner as above.

(b) Directly from the definition, prove that $\lim_{x\to-4} \frac{x}{4+x} = \pm\infty$.

Solution: Let $N > 0$ and set $\delta \leq \underline{1, 3/N}$. Then if $0 < |x+4| < \delta$, we have

$$\left|\frac{x}{4+x}\right| = \frac{|x|}{|x+4|} > \frac{|x|}{\delta}.$$

We observe that $|x| = |x+4-4| \geq 4 - |x+4| \geq 4 - \delta \geq 3$, which gives us

$$\left|\frac{x}{4-x}\right| = \frac{|x|}{|x-4|} > \frac{|x|}{\delta} \geq \frac{3}{\delta} > \frac{3}{3/N} = N.$$

Problem 5.

(a) Directly from the definition, prove that $\lim_{x \to +\infty} x^2 + x + 1 = +\infty$.

Solution: Let $N > 0$ and set $M = \sqrt{N}$. Then if $x > M$, we have

$$x^2 + x + 1 > M^2 + M + 1 = N + \sqrt{N} + 1 > N.$$

(b) Directly from the definition, prove that $\lim_{x \to -2} \frac{x}{(x+2)^2} = -\infty$.

Solution: Let $N > 0$, and set $\delta \leq \underline{1, 1/\sqrt{N}}$. Then if $0 < |x + 2| < \delta$, then

$$\begin{aligned}
(x+2)^2 &< \delta^2 \leq 1/N \\
\frac{1}{(x+2)^2} &> \frac{1}{\delta^2} \geq N \\
x = (x+2) - 2 &< \delta - 2 \leq 1 - 2 = -1 \\
\frac{x}{(x+2)^2} &< -N \qquad \text{(sign flips because } -1 < 0\text{).}
\end{aligned}$$