Math 114 Test 1 Solutions

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Problem 1.

(a) Directly from the definition of a limit, compute with proof $\lim_{x \to 1} 4x + 3$

Solution: We guess 7.

Let $\epsilon > 0$ and let $\delta \leq \underline{\epsilon/4}$. Then if $0 < |x - 1| < \delta$ we have

$$|4x + 3 - 7| = |4x - 4| = 4|x - 1| < 4\delta \leq 4\epsilon/4 = \epsilon.$$

(b) Directly from the definition, compute with proof $\lim_{x \to -3} \frac{x}{x+2}$.

Solution: We guess 3.

Let $\epsilon > 0$ and set $\delta \leq \underline{1/2, \epsilon/4}$. Then if $0 < |x + 3| < \delta$ then

$$\left| \frac{x}{x+2} - 3 \right| = \left| \frac{x}{x+2} - \frac{3x+6}{x+2} \right| = \left| \frac{-2x-6}{x+2} \right| = \frac{2|x+3|}{|x+2|} < \frac{2\delta}{|x+2|}.$$

To handle the bottom, we observe $|x + 2| = |x + 3 - 1| \geq |1| - |x + 3| > 1 - \delta \geq 1/2$. Thus we have

$$\left| \frac{x}{x+2} - 3 \right| < \frac{2\delta}{|x+2|} < \frac{2\delta}{1/2} = 4\delta \leq 4\epsilon/4 = \epsilon.$$

Problem 2.

Let

$$f(x) = \begin{cases} 1 & x < 3 \\ 2 & x > 3 \end{cases}$$

(a) Directly from the definition, compute with proof $\lim_{x \to 0} f(x)$.

Solution: We guess 1.

Let $\epsilon > 0$ and set $\delta = \underline{3}$. Then if $0 < |x| < \delta$, we see that $x < 3$, so $f(x) = 1$, and thus we have

$$|f(x) - 1| = |1 - 1| = 0 < \epsilon.$$

(b) Directly from the definition of a limit, prove that $\lim_{x \to 3} f(x)$ does not exist.

Solution: Set $\epsilon = 1/2$ and suppose $\delta > 0$. Suppose $\lim_{x \to 3} f(x) = L$. Then set $x_1 = 3 + \delta/2, x_2 = 3 - \delta/2$, and we have

$$\begin{aligned} \epsilon &> |f(x_1) - L| = |f(3 + \delta/2) - L| = |2 - L| \\ \epsilon &> |f(x_2) - L| = |f(3 - \delta/2) - L| = |1 - L| \\ 2\epsilon &> |L - 2| + |1 - L| \geq |L - 2 + 1 - L| = |-1| = 1 \end{aligned}$$

Since $\epsilon = 1/2$ this gives us $1 > 1$, which is a contradiction.

Problem 3.
Let

$$g(x) = \begin{cases} 5x+4 & x > 4 \\ 2x+2 & x < 4 \end{cases}$$

(a) Directly from the definition, compute with proof $\lim_{x\to 5} g(x)$.

Solution: Guess 29.

Let $\epsilon > 0$ and set $\delta \le \underline{1, \epsilon/5}$. Then if $0 < |x-5| < \delta$ we have $x > 4$ and thus $g(x) = 5x+4$. So we compute

$$|g(x) - 29| = |5x+4-29| = |5x-25| = 5|x-5| < 5\delta \le 5\epsilon/5 = \epsilon.$$

Thus the limit is 29.

(b) Directly from the definition of a limit, prove that $\lim_{x\to 4} g(x)$ does not exist.

Solution:

Suppose $\lim_{x\to 4} g(x) = L$. Set $\epsilon = \underline{7}$ and let $\delta > 0$. Let $x_1 = 4 - \delta/2$ and $x_2 = 4 + \delta/2$. Then we have

$$\begin{aligned} \epsilon &> |g(x_1) - L| = |2(4-\delta/2)+2-L| = |10-\delta-L| = |L+\delta-10| \\ \epsilon &> |g(x_2) - L| = |5(4+\delta/2)+4-L| = |24+5\delta/2-L| \\ 2\epsilon &> |L+\delta-10| + |24+5\delta/2-L| \ge |14+7\delta/2| = 14+7\delta/2 > 14. \end{aligned}$$

Since $\epsilon = 7$ this gives us $14 > 14$, which impossible. So no such limit exists.

Problem 4. (a) Directly from the definition, prove that $\lim_{x\to 1} \dfrac{1}{x-1} = \pm\infty$.

Solution:

Let $N > 0$ and set $\delta = \underline{1/N}$. Then if $0 < |x-1| < \delta$ we have

$$\left|\frac{1}{x-1}\right| = \frac{1}{|x-1|} > \frac{1}{\delta} = \frac{1}{1/N} = N.$$

(b) Directly from the definition, prove that $\lim_{x\to -\infty} 1 - x = +\infty$.

Solution: Let $N > 0$ and set $M \ge N-1$. Then if $x < -M$ we have

$$1 - x > 1 + M = 1 + N - 1 = N.$$

Problem 5.

(a) Directly from the definition, prove that $\lim_{x\to 3} \dfrac{x}{(x-3)^2} = +\infty$.

Solution: Let $N > 0$ and set $\delta \le \underline{2, 1/\sqrt{N}}$. Then if $0 < |x-3| < \delta$, we see that $x - 3 > -\delta$, so $x > 3 - \delta \ge 1$, and we have

$$\frac{x}{(x-3)^2} > \frac{x}{\delta^2} > \frac{3-\delta}{\delta^2} \ge \frac{1}{\delta^2} \ge \frac{1}{(1/\sqrt{N})^2} = N.$$

(b) Directly from the definition, prove that $\lim_{x\to +\infty} \dfrac{x}{x+4} = 1$.

Solution: Let $\epsilon > 0$ and set $M = \underline{4/\epsilon - 4}$. Then if $x > M$ we have

$$\left|\frac{x}{x+4} - 1\right| = \left|\frac{-4}{x+4}\right| = \frac{4}{x+4} < \frac{4}{M+4} \le \frac{4}{(4/\epsilon - 4) + 4} = \epsilon.$$