Math 114 Practice Test 2 Solutions

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Problem 1.

Compute the following limits, showing each step and naming each limit law you use.

(a)

$$\lim_{x \to 1} (x+1)^3 \frac{x-4}{x}$$

Solution:

$$\begin{aligned}
\lim_{x \to 1} (x+1)^3 \frac{x-4}{x} &= \lim_{x \to 1} (x+1)^3 \lim_{x \to 1} \frac{x-4}{x} && \text{Products} \\
&= \lim_{x \to 1} (x+1)^3 \frac{\lim_{x \to 1} x - 4}{\lim_{x \to 1} x} && \text{Quotients} \\
&= \left(\lim_{x \to 1} x + 1\right)^3 \frac{\lim_{x \to 1} x - 4}{\lim_{x \to 1} x} && \text{Exponents} \\
&= \left(\lim_{x \to 1} x + \lim_{x \to 1} 1\right)^3 \frac{\lim_{x \to 1} x - \lim_{x \to 1} 4}{\lim_{x \to 1} x} && \text{Additivity} \\
&= \left(1 + \lim_{x \to 1} 1\right)^3 \frac{1 - \lim_{x \to 1} 4}{1} && \text{Identity} \\
&= (1+1)^3 \frac{1-4}{1} && \text{Constants} \\
&= -24 && \text{Algebra}
\end{aligned}$$

(b)

$$\lim_{x \to -2} 2 \frac{x^2+5x+6}{x+2}$$

Solution:

$$\begin{aligned}
\lim_{x \to -2} 2 \frac{x^2+5x+6}{x+2} &= 2 \lim_{x \to -2} \frac{x^2+5x+6}{x+2} && \text{Scalars} \\
&= 2 \lim_{x \to -2} \frac{(x+2)(x+3)}{x+2} && \text{Algebra} \\
&= 2 \lim_{x \to -2} x+3 && \text{Almost Identical Functions} \\
&= 2 \left(\lim_{x \to -2} x + \lim_{x \to -2} 3\right) && \text{Additivity} \\
&= 2 \left(-2 + \lim_{x \to -2} 3\right) && \text{Identity} \\
&= 2(-2+3) && \text{Constants} \\
&= 2 && \text{Algebra}
\end{aligned}$$

Problem 2.

Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a) $\lim_{x\to 3^-} f(x)$ where

$$f(x) = \begin{cases} x+5 & x<3 \\ x^2-2 & x>3 \end{cases}$$

Solution: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x+5 = 8$.

(b) $\lim_{x\to -1} \dfrac{x^2-2x-3}{2x+2} =$

Solution:

$$\lim_{x\to -1} \frac{x^2-2x-3}{2x+2} = \lim_{x\to -1} \frac{(x-3)(x+1)}{2(x+1)} = \lim_{x\to -1} \frac{x-3}{2} = -2.$$

(c) $\lim_{x\to 5} \dfrac{\sqrt{x-1}-2}{x-5} =$

Solution:

$$\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x-5} = \lim_{x\to 5} \frac{x-1-4}{(x-5)(\sqrt{x-1}+2)} = \lim_{x\to 5} \frac{1}{\sqrt{x-1}+2} = \frac{1}{4}.$$

(d) $\lim_{x\to 0} \dfrac{\sin(3x)\sin(x)}{x^2} =$

Solution:

$$\lim_{x\to 0} \frac{\sin(3x)\sin(x)}{x^2} = \lim_{x\to 0} \frac{\sin(3x)}{x}\frac{\sin(x)}{x} = \lim_{x\to 0} 3\frac{\sin(3x)}{3x}\frac{\sin(x)}{x} = 3\cdot 1\cdot 1 = 3.$$

by the small angle approximation.

Problem 3. Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a) $\lim_{x\to 2} \dfrac{x^2+3x+2}{x-2} =$

Solution: The limit of the top is $2^2+3\cdot 2+2 = 12$ and the limit of the bottom is zero, so the limit is $\pm\infty$. Since the bottom can be either positive or negative, we can't be more specific.

(b) $\lim_{x\to +\infty} \dfrac{x^2+5x+1}{\sqrt{4x^4-4x+2}} =$

Solution:

$$\lim_{x\to +\infty} \frac{x^2+5x+1}{\sqrt{4x^4-4x+2}} = \lim_{x\to +\infty} \frac{1+5/x+1/x^2}{\sqrt{4-4/x^3+2/x^4}} = \frac{1}{\sqrt{4}} = \frac{1}{2}.$$

(c) Using the Squeeze Theorem, show that

$$\lim_{x\to -1} (x+1)^2 \sin\left(\frac{1}{x+1}\right) = 0.$$

Solution: We know that $-1 \le \sin(a) \le 1$ for any a, and so in particular $-1 \le \sin(1/(x+1)) \le 1$. Then we have

$$-1 \le \sin\left(\frac{1}{x+1}\right) \le 1$$

$$-(x+1)^2 \le (x+1)^2 \sin\left(\frac{1}{x+1}\right) \le (x+1)^2$$

We can compute that $\lim_{x\to -1} -(x+1)^2 = 0$, and similarly $\lim_{x\to-1}(x+1)^2 = 0$. So by the squeeze theorem, we see that $\lim_{x\to-1}(x+1)^2 \sin\left(\frac{1}{x+1}\right) = 0$ as desired.

Problem 4. (a) Show that there is some number c with $0 < c < \pi/2$ such that $\cos(c) = \pi/4$. **Solution:** We know from class that the function cos is continuous. $\cos(0) = 1$ and $\cos(\pi/2) = 0$, and since $0 < \pi < 4$ we can tell that $0 < \pi/4 < 1$. Thus by the intermediate value theorem, there must be some c between 0 and $\pi/2$ such that $\cos(c) = \pi/4$.

(b) Let

$$g(x) = \begin{cases} x^2 - 2 & x < -2 \\ \sqrt{x+6} & x > -2 \end{cases}$$

If possible, define an extension of g that is continuous at all real numbers. If it is not possible, explain why.

Solution: g is continuous everywhere except at -2 by algebra. We compute

$$\lim_{x\to-2^-} g(x) = \lim_{x\to-2^-} x^2 - 2 = 2$$
$$\lim_{x\to-2^+} g(x) = \lim_{x\to-2^+} \sqrt{x+6} = 2$$

So we can define an extension

$$g(x) = \begin{cases} x^2 - 2 & x < -2 \\ \sqrt{x+6} & x > -2 \\ 2 & x = 2 \end{cases}$$

which is continuous at all reals.

Problem 5. Compute the following derivatives using only the definition of derivative.

(a) Derivative of $f(x) = x^3 + 1$ at $x = 1$.

Solution:

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{(1+h)^3 + 1 - 2}{h} \\ &= \lim_{h\to 0} \frac{1 + 3h + 3h^2 + h^3 + 1 - 2}{h} \\ &= \lim_{h\to 0} \frac{3h + 3h^2 + h^3}{h} = \lim_{h\to 0} 3 + 3h + h^2 = 3. \end{aligned}$$

(b) Derivative of $g(x) = \sqrt{x+1}$ at $x = 3$.

Solution:

$$\begin{aligned} g'(3) &= \lim_{h\to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} \\ &= \lim_{h\to 0} \frac{4 + h - 4}{h(\sqrt{4+h} + \sqrt{4})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{4+h} + 2)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{4+h} + 2} = \frac{1}{4}. \end{aligned}$$