Math 322 Fall 2017
Number Theory HW 8
Due Friday, November 3

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem.

($\star$) **Starred Problem:** Let k be a fixed natural number. Show that the equation $\tau(n) = k$ has infinitely many solutions.

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. Calculate (noting that these are factorials!)
   (a) $\phi(10!)$
   (b) $\sigma(10!)$
   (c) $\tau(20!)$

2. We say an integer n is k-perfect if $\sigma(n) = kn$. (Thus a perfect number is a 2-perfect number).
   (a) Show that $120 = 2^3 \cdot 3 \cdot 5$ is 3-perfect.
   (b) Find all 3-perfect numbers of the form $n = 2^k \cdot 3 \cdot p$ for $p > 3$ an odd prime. (Hint: express p as a ratio of other integers.)

3. Suppose $a^p - 1$ is prime. Prove that either $a \leq 2$ or $p = 1$.

4. (a) Determine with proof whether M_{17} is prime. (Feel free to use a calculator).
   (b) Find a factor of $2^{91} - 1$.

5.

   Definition 0.1. If f, g are arithmetic functions, we define the *Dirichlet convolution* of f and g to be
   $$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

We define the function $\iota(n)$ by

$$\iota(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

We say g is the inverse of f (under Dirichlet convolution) if $f * g = \iota$.

(a) Show that $f * g = g * f$; that is, Dirichlet convolution is commutative.

(b) Show that $\iota(n)$ is multiplicative.

(c) Show that $\iota * f = f$ for any arithmetic function f.

6. Prove that if f and g are multiplicative functions, then so is $f * g$.