Math 322 Fall 2017
Number Theory HW 8 Solutions
Due Friday, November 3

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem.

($\star$) **Starred Problem:** Let k be a fixed natural number. Show that the equation $\tau(n) = k$ has infinitely many solutions.

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. Calculate (noting that these are factorials!)

   (a) $\phi(10!)$

   Solution: We see that

$$\begin{aligned} 10! &= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \\ &= 2 \cdot 5 \cdot 3^2 \cdot 2^3 \cdot 7 \cdot 2 \cdot 3 \cdot 5 \cdot 2^2 \cdot 3 \cdot 2 \\ &= 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \end{aligned}$$

   so we compute

$$\begin{aligned} \phi(10!) &= \phi(2^8) \cdot \phi(3^4) \cdot \phi(5^2) \cdot \phi(7) \\ &= 2^7(2-1)3^3(3-1)5^1(5-1)7^0(7-1) \\ &= 2^7 \cdot 3^3 \cdot 2 \cdot 5 \cdot 2^2 \cdot 2 \cdot 3 \\ &= 2^{11} \cdot 3^4 \cdot 5 = 829,440. \end{aligned}$$

   (b) $\sigma(10!)$

   Solution: We have

$$\begin{aligned} \sigma(10!) &= \sigma(2^8) \cdot \sigma(3^4) \cdot \sigma(5^2) \cdot \sigma(7) \\ &= \frac{2^9-1}{2-1} \cdot \frac{3^5-1}{3-1} \cdot \frac{5^3-1}{5-1} \cdot \frac{7^2-1}{7-1} \\ &= 511 \cdot 121 \cdot 31 \cdot 8 = 15,334,088. \end{aligned}$$

(c) $\tau(20!)$

Solution: We have

$$\begin{aligned}\tau(20!) &= \tau(2^{18})\cdot\tau(3^8)\cdot\tau(5^4)\cdot\tau(7^2)\cdot\tau(11^1)\cdot\tau(13^1)\cdot\tau(17^1)\cdot\tau(19)^1\\ &= 19\cdot 9\cdot 5\cdot 3\cdot 2\cdot 2\cdot 2\cdot 2 = 41040.\end{aligned}$$

2. We say an integer n is k-perfect if $\sigma(n) = kn$. (Thus a perfect number is a 2-perfect number).

(a) Show that $120 = 2^3\cdot 3\cdot 5$ is 3-perfect.

Solution: We compute

$$\sigma(120) = \sigma(2^3)\sigma(3)\sigma(5) = \frac{2^4-1}{2-1}\cdot\frac{3^2-1}{3-1}\cdot\frac{5^2-1}{5-1} = 15\cdot 4\cdot 6 = 360 = 3\cdot 120.$$

Thus by definition 120 is 3-perfect.

(b) Find all 3-perfect numbers of the form $n = 2^k\cdot 3\cdot p$ for $p > 3$ an odd prime. (Hint: express p as a ratio of other integers.)

Solution: Suppose $n = 2^k\cdot 3\cdot p$ is 3-perfect. Then we have

$$\begin{aligned}3n = \sigma(n) &= \sigma(2^k)\cdot\sigma(3)\cdot\sigma(p)\\ &= \frac{2^{k+1}-1}{2-1}\cdot\frac{3^2-1}{3-1}\cdot\frac{p^2-1}{p-1}\\ 2^k\cdot 3^2\cdot p &= (2^{k+1}-1)\cdot 4\cdot(p+1)\\ 2^k3^2p &= (2^{k+3}-4)(p+1) = (2^{k+3}-4)p + 2^{k+3}-4\\ p(2^k3^2 - 2^{k+3}+4) &= 2^{k+3}-4\\ p(2^k+4) &= 2^{k+3}-4\\ p(2^{k-2}+1) &= 2^{k+1}-1\end{aligned}$$

So we only have a 3-perfect number when $p = \frac{2^{k+1}-1}{2^{k-2}+1}$ is a prime integer. But note this quantity is always less than 8, so the only possible candidates are $p = 5$ and $p = 7$.

When $p = 5$ then we have

$$\begin{aligned}5\cdot 2^{k-2}+5 &= 2^{k+1}-1\\ 6 &= 2^{k+1} - 52^{k-2} = 2^{k-2}(8-5) = 3\cdot 2^{k-2}\\ 2 &= 2^{k-2}\end{aligned}$$

and thus $k = 3$, which is the example from part (a).

When $p = 7$, then we have

$$\begin{aligned}7\cdot 2^{k-2}+7 &= 2^{k+1}-1\\ 8 &= 2^{k+1} - 7\cdot 2^{k-2} = 2^{k-2}(8-1) = 2^{k-2}\end{aligned}$$

And thus $k-2 = 3$ and $k = 5$. So the only other 3-perfect number of this form is $2^5\cdot 3\cdot 7 = 672$.

3. Suppose $a^p - 1$ is prime. Prove that either $a \leq 2$ or $p = 1$.

   Solution: Suppose $a > 1$ and $a^p - 1$ is prime. We see that $a \equiv 1 \mod a-1$, so $a^p \equiv 1 \mod a-1$, so $(a-1)|a^p - 1$. Since $a^p - 1$ is prime, we must have either $a - 1 = a^p - 1$ hence $p = 1$, or $a - 1 = 1$ hence $a = 2$.

4. (a) Determine with proof whether M_{17} is prime. (Feel free to use a calculator).

   Solution: We know that if M_{17} has a prime factor, it must be of the form $34k+1$. We only need to check numbers less than $\sqrt{M_{17}} \approx 362$. The numbers of this form are $35, 69, 103, 137, 171, 205, 239, 273, 307, 341$. Of these, $103, 137, 239, 307$ are prime.

   We compute

$$(2^{17} - 1)/103 = 1272.53 \qquad (2^{17} - 1)/137 = 956.723$$
$$(2^{17} - 1)/239 = 548.414 \qquad (2^{17} - 1)/307 = 426.941$$

   and thus M_{17} is prime.

   (b) Find a factor of $2^{91} - 1$.

   Solution: We know that $91 - 7 \cdot 13$, so $2^7 - 1 = 127$ and $2^{13} - 1 = 8191$ are both factors of $2^{91} - 1$.

5.

Definition 0.1. If f, g are arithmetic functions, we define the *Dirichlet convolution* of f and g to be

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

We define the function $\iota(n)$ by

$$\iota(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

We say g is the inverse of f (under Dirichlet convolution) if $f * g = \iota$.

(a) Show that $f * g = g * f$; that is, Dirichlet convolution is commutative.

   Solution:

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d) = \sum_{de=n} f(d)g(e)$$
$$= \sum_{e|n} g(e)f(n/e) = (g * f)(n).$$

(b) Show that $\iota(n)$ is multiplicative.

   Solution: Let $(m, n) = 1$. Suppose $m = n = 1$; then $\iota(mn) = 1 = 1 \cdot 1 = \iota(m)\iota(n)$. Now suppose without loss of generality that $m > 1$; then $mn > 1$ so $\iota(mn) = 0$, and $\iota(m)\iota(n) = 0 \cdot \iota(n) = 0$.

(c) Show that $\iota * f = f$ for any arithmetic function f.

Solution: Let f be an arithmetic funtion. Then

$$\begin{aligned}(\iota * f)(n) &= \sum_{d|n} \iota(d) f(n/d) = \iota(1) f(n) + \sum_{d|n, d>1} \iota(d) f(n/d) \\ &= f(n) + \sum_{d|n, d>1} 0 \cdot f(n/d) = f(n).\end{aligned}$$

6. Prove that if f and g are multiplicative functions, then so is $f * g$.

Solution: Suppose f, g are multiplicative functions, and let $(m, n) = 1$. Then

$$\begin{aligned}(f * g)(mn) &= \sum_{d|mn} f(d) g(mn/d) = \sum_{d_1|m, d_2|n} f(d_1 d_2) g\left(\frac{m}{d_1}\frac{n}{d_2}\right) \\ &= \sum_{d_1|m, d_2|n} f(d_1) f(d_2) g(m/d_1) g(n/d_2) = \left(\sum_{d_1|m} f(d_1) g(m/d_1)\right)\left(\sum_{d_2|n} f(d_2) g(n/d_2)\right) \\ &= (f * g)(m) \cdot (f * g)(n).\end{aligned}$$