Math 322 Exam 2 Solutions

Instructor: Jay Daigle

Problem 1. (a) Compute $\mu(91)$.

(b) Compute $\mu(9100)$.

(c) Compute $\sigma(36)$.

(d) Compute $\tau(7!)$.

Solution:

(a) $91 = 7 \cdot 13$ so $\mu(91) = (-1)^2 = 1$.

(b) 9100 is divisible by 4 which is a square. So $\mu(9100) = 0$.

(c) $36 = 2^2 3^2$, so $\sigma(36) = \frac{2^3-1}{2-1}\frac{3^3-1}{3-1} = 7 \cdot 13 = 91$.

(d) $\tau(7!) = \tau(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2) = \tau(2^4 \cdot 3^2 \cdot 5 \cdot 7) = (5)(3)(2)(2) = 60$.

Problem 2. (a) Compute $5^{1234} \mod 18$. (Hint: use Euler's Theorem)

(b) Show that $30240 = 2^5 \cdot 3^3 \cdot 5 \cdot 7$ is 4-perfect.

Solution:

(a) We know that $\phi(18) = 3(3-1)(2-1) = 6$. Thus $5^6 \equiv 1 \mod 18$. Then

$$5^{1234} = (5^6)^{205} \cdot 5^4 \equiv 5^4 \equiv (25)^2 \equiv 7^2 \equiv 49 \equiv 13 \mod 18.$$

(b) We compute that

$$\sigma(30240) = \frac{2^6-1}{2-1}\frac{3^4-1}{3-1}\frac{5^2-1}{5-1}\frac{7^2-1}{7-1} = (63)(40)(6)(8) = 2^7 \cdot 3^3 \cdot 5 \cdot 7 = 120960 = 4 \cdot 30240.$$

Problem 3. Show that $6601 = 7 \cdot 23 \cdot 41$ is a Carmichael number.

Solution: Suppose $(b, 6601) = 1$. Then

$$\begin{aligned} b^6 &\equiv 1 \mod 7 & (b^6)^{1100} &\equiv 1 \mod 7 \\ b^{22} &\equiv 1 \mod 23 & (b^{22})^{300} &\equiv 1 \mod 23 \\ b^{40} &\equiv 1 \mod 41 & (b^{40})^{165} &\equiv 1 \mod 41 \end{aligned}$$

Thus by the Chinese Remainder Theorem, $b^{6600} \equiv 1 \mod 7 \cdot 23 \cdot 41 = 6601$. Thus 6601 is pseudoprime to any base it is relatively prime to, and thus by definition is a Carmichael number.

Problem 4. (a) Find (with proof) every number n with $\phi(n) = 6$.

(b) Find (with proof) every number n with $\phi(n) = 9$.

Solution:

1. Suppose p is a prime factor of n. Then $\phi(p)|6$ so $\phi(p) = 1, 2, 3, 6$. Thus $p = 2, 3, 4, 7$, and 4 isn't prime, so we can only have $p = 2, 3, 7$. Thus we take $n = 2^a 3^b 7^c$.

   Then we have $\phi(n) = 2^{a-1} \cdot 2 \cdot 3^{b-1} \cdot 6 \cdot 7^{c-1}$, and we immediately see that $c \leq 1$.

   If $c = 0$ then we have $n = 2^a 3^b$. Assume $a, b > 0$. Then $6 = \phi(n) = 2^{a-1} \cdot 2 \cdot 3^{b-1}$ so we must have $b = 2$, which gives $a = 1$. So $\phi(18) = 6$.

   If $c = 0, a = 0$ then we have $n = 3^b$ and $\phi(n) = 2 \cdot 3^{b-1}$, which again gives $b = 2$. Thus $\phi(9) = 6$.

   If $c = 0, b = 0$, then $n = 2^a$ and $6 = \phi(n) = 2^{a-1}$ which isn't possible.

   Now if $c > 0$, we have $6 = \phi(n) = \phi(2^a 3^b)\phi(7^c) = \phi(2^a 3^b) \cdot 6 \cdot 7^{c-1}$. Thus $c = 1$ and we have $\phi(2^a 3^b) = 1$. This implies that either $2^a 3^b = 1$ or $2^a 3^b = 2$, so we have $\phi(7) = 6$ and $\phi(14) = 6$.

   Thus the final set of solutions is $n \in \{7, 9, 14, 18\}$.

2. We know that $\phi(n)$ is even unless $n = 1$ or $n = 2$, in which case $\phi(n) = 1$. So there are no n with $\phi(n) = 9$.

Problem 5. If n is square-free, show that

$$\sum_{d|n} (\mu(d))^2 \phi(d) = n.$$

Solution:
Suppose n is square-free. Then d is square-free for any $d|n$, so $\mu(d) = \pm 1$ and $(\mu(d))^2 = 1$. Thus

$$n = \sum_{d|n} \phi(d) = \sum_{d|n} (\mu(d))^2 \phi(n/d)$$

Note: $\sum_{d|n} \mu(d)^2 \phi(d) \neq \left(\sum_{d|n} \mu(d)^2\right)\left(\sum_{d|n} \phi(d)\right)$. This is easy to see if you write it out with actual numbers:

$$\sum_{d|6} \mu(d)^2 \phi(d) = \mu(1)^2\phi(1) + \mu(2)^2\phi(2) + \mu(3)^2\phi(3) + \mu(6)^2\phi(6)$$

$$\begin{aligned}
\left(\sum_{d|6} \mu(d)^2\right)\left(\sum_{d|6} \phi(d)\right) &= (\mu(1)^2 + \mu(2)^2 + \mu(3)^2 + \mu(6)^2)(\phi(1) + \phi(2) + \phi(3) + \phi(6)) \\
&= \mu(1)^2\phi(1) + \mu(2)^2\phi(1) + \mu(3)^2\phi(1) + \mu(6)^2\phi(1) \\
&\quad + \mu(2)^2\phi(1) + \mu(2)^2\phi(2) + \mu(3)^2\phi(2) + \mu(6)^2\phi(2) \\
&\quad + \mu(3)^2\phi(1) + \mu(2)^2\phi(3) + \mu(3)^2\phi(3) + \mu(6)^2\phi(3) \\
&\quad + \mu(6)^2\phi(1) + \mu(2)^2\phi(6) + \mu(3)^2\phi(6) + \mu(6)^2\phi(6)
\end{aligned}$$