Coding Theory

Jay Daigle

Occidental College

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What is coding?

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- Ideally gain extra features.

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- Thoughts into strings of letters
- Letters into numbers
- Pictures into binary strings

Types of Codes

- Numeric encoding

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- First developed for formal logic. Gödel Incompleteness Theorem.

Gödel Numbers

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$$\text{zzz} \mapsto 2^2 53^2 55^2 5 = 847288609443000000000000000000000000$$

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- Lossy encoding: C is not injective.

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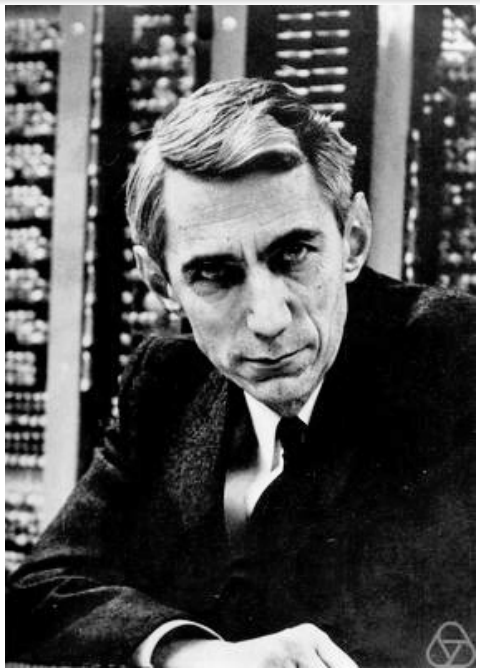

Claude Shannon (1916 – 2001)

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symbol i	p_i	$\lceil -\log_2 p_i \rceil$	$\sum_{n=0}^{i-1} p_n$	in binary	codeword
1	.36	2	0.0	.0000	00
2	.18	3	.36	.0101	010
3	.18	3	.54	.1000	100
4	.12	4	.72	.1011	1011
5	.09	4	.84	.1101	1101
6	.07	4	.93	.1110	1110

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Arithmetic Coding

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- In practice, send one number in that interval
- Optimal under slightly different conditions

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Eighth bit is "parity bit". Can detect single-bit error in each byte.

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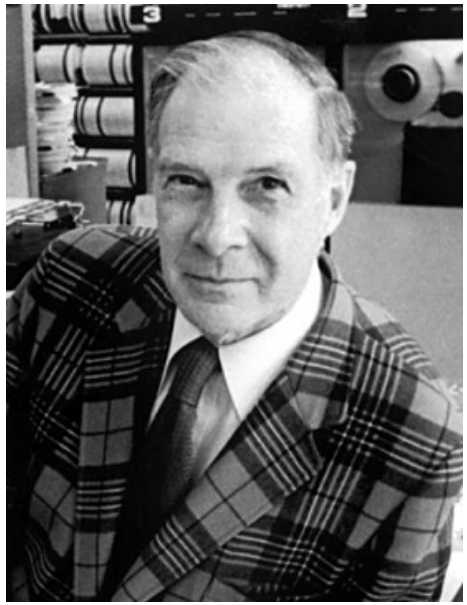

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Invented to fix problems with a punched-card reader at Bell Labs.