Precalculus Review

Important Things to Review

Functions

Recall that a *function* is a rule that takes an input and assigns a specific output. Note that a function always gives exactly one output, and always gives the same output for a given input. Here we remember some facts about common functions.

Polynomials: You should remember the quadratic formula, which says that if $ax^2 + bx + c = 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is also useful to recall that

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a+b)(a-b) = a^2 - b^2$
- $(a^2 + ab + b^2)(a-b) = a^3 - b^3$.

Rational functions are the ratio of two polynomials.

Trigonometric functions: In this course we will *always* use radians, because they are unitless and thus easier to track (especially when using the chain rule). Useful facts include:

- The most important trigonometric identity, and really the only one you probably need to remember, is $\cos^2(x) + \sin^2(x) = 1$.
- From this you can derive the fact that $1 + \tan^2(x) = \sec^2(x)$.
- $\sin(-x) = -\sin(x)$. We call functions like this "odd".
- $\cos(-x) = \cos(x)$. We call functions like this "even."
- $\sin(x + \pi/2) = \sin(\pi/2 - x) = \cos(x)$
- A fact that we will probably use exactly twice is the sum of angles formula for sine: $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.
- Similarly, $\cos(x+y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

The Absolute Value Function

The absolute value function $|x|$ will be important to us. It is even. We can think of it as a piecewise function

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

Or we can think of it as $|x| = \sqrt{x^2}$; both of these definitions can be useful.

The absolute value function is important because it is used to think about distances, which will be a core idea in this course.

Definition 0.1. The *distance* between two numbers x and y is the absolute value of their difference $|x - y|$.

Example 0.2. The distance between 3 and 5 is $|3 - 5| = 2$. (Notice this is the same as the distance $|5 - 3| = 2$ between 5 and 3).

The distance between 2 and -2 is $|2 - (-2)| = 4$. The distance between -2 and 2 is $|-2-2| = 4$.

We know the shortest distance between two points is a straight line; this implies that the distance between x and y is no larger than the distance from x to z plus the distance from z to y–that is, we can never make our path shorter by detouring through another point z. This leads us to formulate a very important rule known as the *Triangle Inequality*.

Fact 0.3 (Triangle Inequality)**.** *If x and y are real numbers, then $|x + y| \leq |x| + |y|$.*

Geometrically, we see that $|x + y| = |x - (-y)|$ is the distance between x and $-y$. But $|x| = |x - 0|$ is the distance between x and 0, and $|y| = |0 - (-y)|$ is the distance between 0 and $-y$. Thus, since the distance from x to $-y$ is no bigger than the distance from x to 0 plus the distance from 0 to $-y$, we have $|x + y| \leq |x| + |y|$.

We can also interpret this statement algebraically. Algebraically, we see that the x and y can cancel each other out *inside* the absolute value signs, but not *outside* them. (That is, $|5 + -3| = 2$ but $|5| + |-3| = 8$).

Example 0.4. $8 = |3 + 5| \leq |3| + |5| = 8$.

$4 = |6 - 2| \leq |6| + |-2| = 8$. (Note that we're treating $6 - 2$ as $6 + (-2)$ here).

Example 0.5. If $|f(x)| \leq 2$ and $|g(x)| \leq |x + 1|$ then $|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq 2 + |x + 1|$.

In contrast, if $|f(x)| \geq 2$ and $|g(x) \leq x^2$ then we can say $|f(x)+g(x)| \leq |f(x)|+|g(x)| \leq |f(x)| + x^2$. We can't get rid of the $|f(x)|$ term because we don't have an *upper* bound on it, only a lower bound.

Remark 0.6. We call this the "Triangle Inequality" because, when generalized to two dimensions, it tells us that the length of one side of a triangle is no longer than the sum of the lengths of the other two sides. In this one-dimensional case, the triangle is extremely narrow and has all three points on the same line, which makes it look rather less triangular.

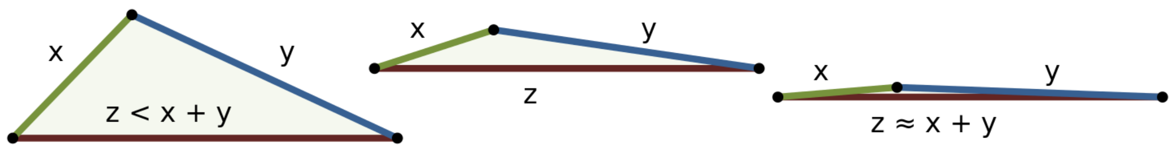

Figure 0.1: The triangle equality in two dimensions, shrinking towards one.
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We can also derive the *inverse triangle inequality*:

Lemma 0.7. *If x and y are real numbers, then $|x + y| \geq |x| - |y|$.*

Proof. The algebraic proof is short, and takes advantage of the fact that $|y| = |-y|$ for any number y.

$$|x| = |x + y - y| \leq |x + y| + |-y| \qquad \text{by the triangle inequality}$$
$$|x| - |y| \leq |x + y| + |-y| - |y| = |x + y|.$$

□

Geometrically, this tells us that any side of a triangle is longer than the difference of the lengths of the other two sides–which should make sense, since it needs to be long enough to connect them.

Example 0.8. $8 = |3 + 5| \geq |3| - |5| = -2$. Also, $8 = |5 + 3| \geq |5| - |3| = 2$.

$4 = |6 - 2| \geq |6| - |-2| = 4$. Also, $4 = |2 - 6| \geq |2| - |-6| = -4$.

Example 0.9. If $|f(x)| \geq 2$ and $|g(x)| \leq |x + 1|$ then $|f(x) + g(x)| \geq |f(x)| - |g(x)| \geq 2 - |x + 1|$.

In contrast, if $|f(x)| \leq 2$ and $|g(x) \leq x^2$ then we can say $|f(x)+g(x)| \geq |f(x)|-|g(x)| \geq |f(x)| - x^2$. We can't get rid of the $|f(x)|$ term because we don't have a *lower* bound on it, only an upper bound.

Remark 0.10. The important difference between the triangle inequality and the reverse triangle inequality isn't the starting point; they both start with $|x + y|$. But the triangle inequality tells us that $|x+y|$ is smaller than something, while the inverse triangle inequality tells us that $|x + y|$ is bigger than something.

Set and interval notation

We write $\{x : \text{condition}\}$ to represent the set of all numbers x that satisfy some condition. We will sometimes write $\mathbb{R}$ to refer to all the real numbers. We will also refer to various intervals:

$(a, b) = \{x : a < x < b\}$	open interval	$[a, b] = \{x : a \leq x \leq b\}$	closed interval
$[a, b) = \{x : a \leq x < b\}$	half-open interval	$(a, b] = \{x : a < x \leq b\}$	half-open interval

Introduction to functions

Definition 0.11. A *function* is a rule that takes an input and assigns a specific output. Note that a function always gives exactly one output, and always gives the same output for a given input.

In the abstract, a function can take any type of input and give any type of output. In this class we will primarily study functions whose inputs and outputs are all real numbers.

Definition 0.12. The *domain* of a function is the set of possible valid inputs. The *range* or *image* is the set of possible outputs.

Example 0.13.

1. The function $f(x) = x^2$ has all real numbers in its domain, and its image is the set of non-negative real numbers.
2. The function $f(x) = \sqrt{x}$ has all non-negative real numbers as its domain, and non-negative real numbers as its image.
3. The function $f(x) = \frac{1}{x^2-1}$ has all real numbers except 1 and -1 in its domain, and all real numbers greater than zero or less than or equal to -1 in its image. We can write this set as $\{x : x > 0 \text{ or } x \leq -1\}$, or equivalently as $\{x : x > 0\} \cup \{x : x \leq -1\}$ or $(-\infty, -1] \cup (0, +\infty)$.

Remark 0.14. The word "range" is sometimes used to refer to the type of output a function can have; in this context people also use the word "codomain". In this class we will always use "range" to refer to an output a function can actually produce.

Functions can be described many ways: a verbal description, an algebraic rule, a graph, or a list of possible inputs and the corresponding outputs.

Poll Question 0.0.1. What are the domain and range of $f(x) = x^3$?

The domain of the function is all real numbers, since we can cube any number. Less obviously, the range is also all reals: if we cube a negative number, we get a negative number, and if we cube a positive number we get a positive number.

Poll Question 0.0.2. What are the domain and range of $\frac{1}{x-1}$?

The domain is all reals except 1, because we can't divide by zero. (In general, the domain is often "everywhere nothing goes wrong.") The image is all reals except 0, since we can divide 1 by any number except 0 and thus get the reciprocal of any non-zero number.

In other notation, the domain is $\{x : x \neq 1\}$ and the range is $\{x : x \neq 0\}$.

Definition 0.15. A *piecewise function* is a function defined by breaking its domain up into pieces and giving a rule for each piece.

Example 0.16. 1.

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

is a piecewise function, given by the rule "If the input is negative, the output is zero; otherwise the output is 1." The domain is all reals and whose range is $\{0, 1\}$.

2.

$$g(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x \geq 0 \end{cases}$$

is not a function because it does not give a clear output when given 0 as input.

3.

$$h(x) = \begin{cases} x^2 + 1 & x < 0 \\ 3x - 2 & x > 0 \end{cases}$$

is a piecewise function whose domain does not include 0. The domain is $\{x : x \neq 0\}$ and the range is $(-2, +\infty)$.

4.
$$f(x) = \begin{cases} x+2 & x \geq 1 \\ x^2+2 & x \leq 1 \end{cases}$$

This function might concern you since it appears to have two values for 1; but after looking a bit more closely we see that both pieces define $f(x) = 3$ so we're okay. This is a function whose domain is all reals and whose image is $[2, +\infty)$.

Function Catalog

We will now present a list of functions; we should be familiar with these functions, their graphs, and often their domains and images.

1. A *constant function* is given by $f(x) = c$ for some real number c. It's domain is all real numbers, and its range is the set with one point $\{c\}$.

2. A *linear function* is given by $f(x) = ax + b$. Its domain and range are both all real numbers.

3. A *polynomial function* is given by $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where n is some positive integer and the a_i are all real numbers. A polynomial is a sum of terms, where each term is some real number multiplied by x raised to a positive integer power.

   The domain of any polynomial is all real numbers.

   (3a) A *quadratic polynomial* is a polynomial whose highest term has exponent 2, given by $f(x) = ax^2 + bx + c$. It has image $\{x : x \geq C\}$ or $\{x : x \leq C\}$ for some real number C.

   It will be useful to recall the quadratic formula; if $f(x) = ax^2 + bx + c$ then $f(x) = 0$ precisely when
   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

   (3b) A *cubic polynomial* has 3 as its highest exponent, given by $f(x) = ax^3+bx^2+cx+d$. Its image is all real numbers.

4. A *rational function* is given by the ratio of two polynomial functions (note the similarity between "ratio" and "rational"). Thus a rational function is of the form
$$f(x) = \frac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 + b_1 x + \cdots + b_m x^m}.$$

A rational function has domain all real numbers, except for the finite collection of points where the denominator is zero.

Example 0.17.
- $f(x) = \frac{x^2+1}{x-1}$ is a rational function with domain $\{x : x \neq 1\}$.
- $g(x) = \frac{1}{x^4+7}$ is a rational function with domain all reals, since the denominator is never zero for any real number. (The range is $(0, 1/7]$).

5. The function

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases} = \sqrt{x^2}$$

is well-defined since both rules give the same output for 0. This function is called the *absolute value* of x. The piecewise definition is usually more useful. The domain is all reals, and the image is $[0, +\infty)$; in fact, the point of this function is to "sanitize" all your real number inputs into positive numbers.

We will now discuss the exponential functions.

1. The *n-th root function* is given by $f(x) = x^{1/n}$. $x^{1/n}$ represents the unique positive number y such that $y^n = x$. If n is even then this function has all non-negative numbers in its domain and image; if n is odd then all real numbers are in the domain and image.

2. The *reciprocal function* is given by $f(x) = x^{-1} = \frac{1}{x}$. This function has domain and range $\{x : x \neq 0\}$. It also has the interesting property that $f(f(x)) = x$ for any $x \neq 0$; that is, applying the rule twice gets you back where you started.

3. We can define a general exponential function $f(x) = x^{m/n}$ where m and n are any integers by combining the previous two rules with the rules that
   - $x^a x^b = x^{a+b}$
   - $(x^a)^b = x^{ab}$
   - $x^a y^a = (xy)^a$

Example 0.18. If we wish to calculate $8^{-5/3}$, we can rewrite this as

$$(8^{5/3})^{-1} = ((8^{1/3})^5)^{-1} = (2^5)^{-1} = 32^{-1} = \frac{1}{32}.$$

Poll Question 0.0.3. Compute $27^{-2/3}$.

$$27^{-2/3} = ((27^{1/3})^2)^{-1} = (3^2)^{-1} = 9^{-1} = \frac{1}{9}.$$

Poll Question 0.0.4. What is the domain of $f(x) = \dfrac{x^2-4}{x^2+5x+6}$?

The domain is all reals except where the denominator is zero. $x^2+5x+6 = (x+2)(x+3)$ is zero when $x = -2$ or $x = -3$, so the domain is $\{x : x \neq -2, -3\}$.

Now we discuss the trigonometric functions. In calculus we essentially always use radians. Recall that $\sin(x)$ and $\cos(x)$ are given by the unit circle: if we start from the point $(1, 0)$ and rotate x radians counterclockwise, then our x coordinate will be $\cos(x)$ and our y coordinate will be $\sin(x)$. We can also recall that if θ is the measure of a non-right angle of a right

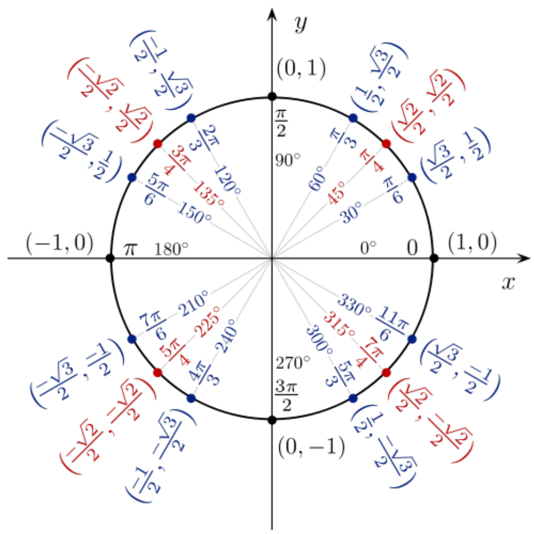

Figure 0.2: The Unit Circle

triangle, then $\sin(\theta)$ is the ratio of the length of the opposite side to the length of the hypotenuse, and $\cos(\theta)$ is the ratio of the length of the adjacent side to the length of the hypotenuse.

There is one important trigonometric identity we must remember, which is that $\sin^2(x) + \cos^2(x) = 1$; this is just the Pythagorean theorem applied to triangles with hypotenuse of length one.

We can see that sin and cos both have domain all reals, and image $[-1, 1]$.

We also have four other trigonometric functions:

1. $\tan(x) = \frac{\sin(x)}{\cos(x)}$ has domain $\{x : x \neq n\pi + \pi/2\}$ since the function isn't defined when $\cos(x) = 0$, and has image all reals.

2. $\cot(x) = \frac{\cos(x)}{\sin(x)}$ has domain $\{x : x \neq n\pi\}$ since the function isn't defined hwen $\sin(x) = 0$, and has image all reals.

3. $\sec(x) = \frac{1}{\cos(x)}$ has domain $\{x : x \neq n\pi + \pi/2\}$ and image $(-\infty, -1] \cup [1, +\infty)$.
4. $\csc(x) = \frac{1}{\sin(x)}$ has domain $\{x : x \neq n\pi\}$ and image $(-\infty, -1] \cup [1, +\infty)$.

The trigonometric functions also have a few important symmetries:

- $\sin(-x) = -\sin(x)$. Functions with this property are called *odd functions*.
- $\cos(-x) = \cos(x)$. Functions with this proprty are called *even functions*.
- $\sin(\pi/2 - x) = \cos(x)$. The sin function is a *reflection* of the cos function around the line $x = \pi/4$.
- $\sin(x + \pi/2) = \cos(x)$. The sin function is a *translation* of the cos function along the x axis.

This leads into our next topic, which is to ask how we can turn some functions into other functions.

Deriving functions from other functions

We can't possibly list every function we will ever use. Instead, let's talk about how to start with a few functions—the ones above—and use them to construct more functions.

Poll Question 0.0.5. What must I do to graph A to get graph B?

Figure 0.3: Left: graph A, Right: graph B

Poll Question 0.0.6. What must I do to graph C to get graph D?

Now we can move on to the main event: various operations we can apply to a function to get a new function.

Assume that c is a positive real number.

We can *shift* the graph of a function up, down, left, or right:

- The graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted up by c units.

Figure 0.4: Left: graph C, Right: graph D

- The graph of $y = f(x) - c$ is the graph of $y = f(x)$ shifted down by c units.
- The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted right by c units.
- The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted left by c units.

Note the perhaps-counterintuitive directions on the last two.

Poll Question 0.0.7. The first graph is the graph of x^2. What is the second graph?

Figure 0.5: The graphs of x^2 and $x^2 - 1$

Answer: $x^2 - 1$. (Since there's no axis labels, $x^2 - c$ would also be reasonable).

Poll Question 0.0.8. What do I need to do to the graph of x^3 to get the graph of $(x + 3)^3$?

Figure 0.6: The graphs of x^3 and $(x + 3)^3$

Answer: shift it to the left by three units.

We can also *stretch* the graph of a function vertically or horizontally.

- The graph of $y = c \cdot f(x)$ is the graph of $y = f(x)$ stretched vertically by a factor of c. Note c can be less than one here, in which case the graph is shrunk.

- The graph of $y = f(x/c)$ is the graph of $y = f(x)$ stretched horizontally by a factor of c. Note again that c can be less than one, in which case the graph is shrunken.

Poll Question 0.0.9. If I stretch the function $\sin(x)$ to be twice as tall, what function do I get?

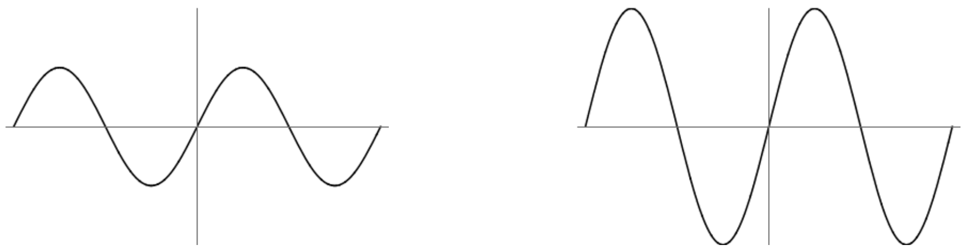

Figure 0.7: The graphs of $\sin(x)$ and $2\sin(x)$

We can also *reflect* a graph about the x axis or y axis (or, with a little creativity, some other axis).

- The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis, that is, flipped top-to-bottom.
- The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected about the y-axis, that is, flipped left-to-right.

Example 0.19. Here is an example of what a function looks like reflected.

Figure 0.8: The graphs of $x^3 + 2x^2$ and $-x^3 + 2x^2$

Figure 0.9: The graphs of $-x^3 - 2x^2$ and $x^3 - 2x^2$

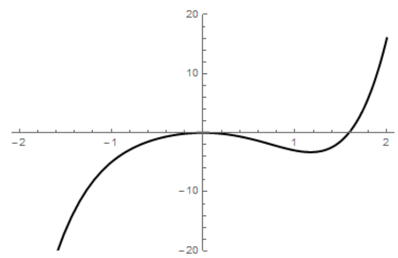

Figure 0.10: The graph of $x^5 - 4x^2$

Poll Question 0.0.10. Figure 0.10 is the graph of $x^5 - 4x^2$. What would the graph of $(x+1)^5 - 4(x+1)^2$ look like? What would the graph of $(2x)^5 - 4(2x)^2$ look like?

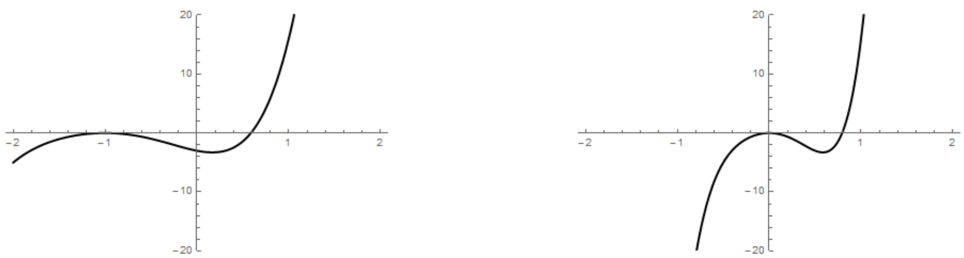

Figure 0.11: The graphs of $(x+1)^5 - 4(x+1)^2$ and $(2x)^5 - 4(2x)^2$

Poll Question 0.0.11. Which of the functions $f(x) = x^2 + 1$, $f(x) = x^3 + 3$, $f(x) = x^4$, $f(x) = x^5 + x$ is even?

Poll Question 0.0.12. Which of the functions $f(x) = x^2 + 1$, $f(x) = x^3 + 3$, $f(x) = x^4$, $f(x) = x^5 + x$ is odd?

In general a polynomial with only even-degree terms will be even, and a polynomial with only odd-degree terms is odd. (Hopefully this will be easy to remember!) A polynomial with both even-degree and odd-degree terms is generally neither even nor odd.

Finally, we can combine two functions.

- The function $f + g$ is defined by $(f + g)(x) = f(x) + g(x)$.
- The function $f \cdot g$ is defined by $(f \cdot g)(x) = f(x)g(x)$.
- The function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$.

This last rule will be very important, and is called *composition of functions*. $f \circ g$ corresponds to putting our input into the function g, and then taking the output and feeding that into the function f. This only makes sense if the image of g is in the domain of f.

Remark 0.20. $f \circ g$ and $g \circ f$ are not the same thing. For instance, if $f(x) = x^2$ and $g(x) = x+1$, then $(f \circ g)(x) = f(x+1) = (x+1)^2 = x^2+2x+1$, but $(g \circ f)(x) = g(x^2) = x^2+1$.

Poll Question 0.0.13. If $f(x) = \sqrt{x}$ and $g(x) = 3x^2$ then what is$(f \circ g)(x)$? What is the domain? What about $(g \circ f)(x)$?

$(f \circ g)(x) = \sqrt{3x^2}$. This is the same as $\sqrt{3}|x|$. The domain is all reals.

$(g \circ f)(x) = 3\sqrt{x}^2$. This is the same as $3|x|$ but the domain is only $[0, +\infty)$ since we can't plug a negative number into f.

Poll Question 0.0.14. Can we write $x^2 + 1$ as the composition of two simple functions?

Answer: Let $f(x) = x^2$ and $g(x) = x + 1$. Then $g(f(x)) = x^2 + 1$

Can we write $\sqrt{x^3 - 1}$ as the composition of three simple functions?

Answer: Let $f(x) = x^3$, $g(x) = x - 1$, and $h(x) = \sqrt{x}$. Then $h(g(f(x))) = \sqrt{x^3 - 1}$.