8 Inverse Functions and Exponentials

8.1 Inverse Functions

Remember we started out by saying that a function is a process: it takes an input and returns an output. Sometimes we want to undo this process. This is in fact a natural question; "What do I have to do if I want to get X" is a pretty common thought process. So our goal is: given a function f, given $f(x)$, can we find x?

Definition 8.1. If f is a function and $(g \circ f)(x) = x$ for every x in the domain of f, then we say g is an *inverse* of f.

Example 8.2.

- If $f(x) = x$ then $g(y) = y$ is an inverse to f.
- If $f(x) = 5x + 3$ then $g(y) = (y - 3)/5$ is an inverse to f.
- If $f(x) = x^3$ then $g(y) = \sqrt[3]{y}$ is an inverse to f.

Remark 8.3. A given function f has at most one inverse—if f has an inverse at all, then that means "for any y, find the x where $f(x) = y$" is a well-defined rule.

If g is an inverse to f, then the domain of g is the image of f and the domain of f is the image of g.

Unfortunately, we can't always find these inverses. For instance, if you know that $x^2 = 9$, you don't know for sure what x is: it could be 3 or -3. Similarly, if you know $\sin(x) = 0$, then x could be $n\pi$ for any integer n. The fundamental problem here is that there are some outputs that are generated by more than one input.

Definition 8.4. A function f is *1-1* or *one-to-one* (or *injective*) if, whenever $f(a) = f(b)$, we know that $a = b$.

Example 8.5. Functions which are 1-1:

- $f(x) = x$. If $f(a) = f(b)$ then $a = b$ by definition.
- $f(x) = x^3$. If $f(a) = f(b)$ then $a^3 = b^3$, and then $(a/b)^3 = 1$ so $a/b = 1$ and $a = b$.
- $f(x) = \sqrt{x}$. If $f(a) = f(b)$ then $\sqrt{a} = \sqrt{b}$ so $|a| = |b|$. But $a, b \geq 0$ since they're in the domain of f, and thus $a = b$.

Functions which are not 1-1:

- $f(x) = x^2$, since $f(-1) = f(1)$.
- $f(x) = |x|$, since $f(-2) = f(2)$.
- $\sin(x)$, since $\sin(0) = \sin(\pi)$.
- $f(x) = 3$, since $f(a) = f(b) = 3$ for any real numbers a and b.

However, we can often force a function to be one-to-one by restricting its domain.

Example 8.6.
- The function $f(x) = x^2$ on the domain $[0, +\infty)$ is 1-1. If $f(a) = f(b)$ then $a^2 = b^2$ so $a = \pm b$. But both $a, b \geq 0$ so $a = b$.
- The function $\sin(x)$ is 1-1 on the domain $[-\pi/2, \pi/2]$. If we look at the unit circle, we see that as x varies from $-\pi/2$ to $\pi/2$, the y coordinate on the unit circle is always increasing, and so never repeats itself.

This might lead us to think graphically about what the idea of 1-1-ness means:

Proposition 8.7 (Horizontal Line Test). *A function f is 1-1 if and only if any horizontal line will intersect its graph in at most one point.*

It's reasonably clear that every function with an inverse must be one-to-one, since otherwise there's not a unique answer to the inverse question. Less obvious is that being 1-1 is enough to be invertible.

Proposition 8.8. *If f is a 1-1 function with domain A and image B, then there is a function f^{-1} with domain B and image A which is an inverse to f.*

We can find this inverse by writing the equation $y = f(x)$ and solving for x as a function of y. Finding an inverse for f is also a good way to prove that f is one-to-one.

Example 8.9. Let $f(x) = x^4$ with domain $(-\infty, 0]$. Then we have $y = x^4 \Rightarrow x = \pm\sqrt[4]{y}$. But we know that $x < 0$ so $x = -\sqrt[4]{y}$, and thus $g(y) = -\sqrt[y]{y}$ is an inverse for f.

Graphically, the graph of f^{-1} looks like the graph of f flipped acroos the line $y = x$, which makes sense, since a point (x, y) on the graph of f should correspond to a point (y, x) on the graph of f^{-1}. In fact, the Horizontal Line Test mentioned earlier is basically the Vertical Line Test applied to the inverse function.

Example 8.10. Take $f(x) = x^3 - x$. This function is clearly not one-to-one, since $f(1) = f(0) = f(-1) = 0$. But we can split it up into intervals where it is one-to-one. Looking at the graph, it seems natural to split it up at the critical points. And this suggests we should use calculus to study our inverse function problem.

8.1.1 Calculus of inverse functions

Now that we understand inverse functions as functions, we'd like to see what calculus can tell us about them.

Proposition 8.11. *If f is one-to-one and continuous at a, then f^{-1} is continuous at $f(a)$.*

If f is one-to-one and continuous, then f^{-1} is continuous.

We'd really like to know about the derivatives of inverse functions. We can work out what they are with some quick sketched arguments, and then can prove the answer rigorously once we know what we're looking for.

First, the argument by "it looks nice in the notation": we can rephrase this theorem as saying that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Second, if we already know that both functions are differentiable, we can use implicit differentiation:

$$\begin{aligned} f^{-1}(f(x)) &= x \\ (f^{-1})'(f(x)) \cdot f'(x) &= 1 \\ (f^{-1})'(f(x)) &= \frac{1}{f'(x)}. \end{aligned}$$

Writing $x = f^{-1}(a)$, or equivalently $a = f(x)$, gives our statement.

Theorem 8.12 (Inverse Function Theorem)**.** *If f is a one-to-one differentiable function, and $f'(f^{-1}(a)) \neq 0$, then $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$.*

Proof. Set $y = f^{-1}(x)$ and $b = f^{-1}(a)$. Then

$$\begin{aligned} (f^{-1})'(a) &= \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a} \\ &= \lim_{y \to b} \frac{y - b}{f(y) - f(b)} \\ &= \lim_{y \to b} \frac{1}{\frac{f(y) - f(b)}{y - b}} \\ &= \frac{1}{f'(b)} = \frac{1}{f'(f(a))}. \end{aligned}$$

□

Example 8.13. Let $f(x) = x^n$ on $[0, +\infty)$; then $f^{-1}(x) = \sqrt[n]{x}$. Our formula gives

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\sqrt[n]{(a)})}$$
$$= \frac{1}{n(\sqrt[n]{(a)})^{n-1}} = \frac{1}{na^{(n-1)/n}} = \frac{1}{n}a^{(1-n)/n} = \frac{1}{n}a^{\frac{1}{n}-1}.$$

Though at first this didn't look like our original answer, it is the same as the formula we had before.

Example 8.14. Let $f(x) = \sqrt[3]{5x^2+7}$. What is $(f^{-1})'(3)$?

Well, we have $(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))}$. We know that $f'(x) = \frac{1}{3}(5x^2+7)^{-2/3} \cdot 10x$, and we can work out that $f(2) = \sqrt[3]{20+7} = 3$ (by plugging in small integers until one works). Thus $f^{-1}(3) = 2$, and so we have

$$(f^{-1})'(3) = \frac{1}{\frac{1}{3}(27)^{-2/3} \cdot 20} = \frac{3 \cdot 9}{20} = \frac{27}{20}.$$

8.2 The exponential and the logarithm

Back in the first weeks of the course, we discussed the exponential functions. It's simple to define x^n when n is a positive integer, as $x \cdot x \cdot \dots \cdot x$. It's now clear that we defined $x^{1/n}$ as the inverse function to x^n, with domain restricted to positive numbers in the case n is even and thus x^n is not one-to-one. But can we make sense of x^r where r is any real number? What would it mean to write $2^{\sqrt{2}}$?

The answer would presumably be between 2 and 4. And also between $2^{1.4}$ and $2^{1.5}$. And between $2^{1.41}$ and $2^{1.42}$. In fact, this is how we will define $2^{\sqrt{2}}$. It turns out that there will be exactly one number greater than $2^1, 2^{1.4}, 2^{1.41}, 2^{1.414}, 2^{1.4142}, \dots$ and less than $2^2, 2^{1.5}, 2^{1.42}, 2^{1.415}, 2^{1.4143}, \dots$.

Definition 8.15. If r is any real number, and a is a positive real number, we define $a^r = \lim_{x \to r} a^x$ for x varying over the rational numbers. We say that a is the *base* and r is the *exponent*.

Remark 8.16. We can't actually raise a negative real number to an irrational power. The limit would vary over x with even denominator, and a^x is not defined if x has even denominator and $a < 0$.

Proposition 8.17. *The exponential function $f_a(x) = a^x$ is well-defined for any r when $a > 0$, and is continuous on all real numbers. Further, it satisfies the exponential laws:*

- $a^{x+y} = a^x a^y$
- $a^{x-y} = \frac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$.

Proposition 8.18. *If* $a > 1$, *then* $\lim_{x\to+\infty} a^x = +\infty$ *and* $\lim_{x\to-\infty} a^x = 0$.

If $0 < a < 1$ *then* $\lim_{x\to+\infty} a^x = 0$ *and* $\lim_{x\to-\infty} a^x = +\infty$.

Proof. Both of these can be seen by considering cases where x is an integer. □

There is a number which we will see works much better as a base for the exponential function than any other. This is the number

$$e = \lim_{x\to 0}(1+x)^{1/x}.$$

It's possible to prove that this limit exists, but not incredibly easy. It happens that $e \approx 2.71828$. We often write exp for the exponential function with base e; that is, $\exp(x) = e^x$.

Remark 8.19. The number e was discovered by Jacob Bernoulli in the context of compound interest. If your interest rate is r and it's compounded n times a year, then the growth rate per year is $(1+\frac{r}{n})^{1/n}$. If the interest is "compounded continuously," your money grows at a rate equal to the limit of this expression as n goes to $+\infty$—which is e. The number was named by Leonhard Euler (hence the "e") when he used it for logarithms.

We'd like to compute the derivative of exp, and also of a^x for a positive real number a. This is a bit difficult to do directly. So we will, as usual, cheat.

8.2.1 Logarithms

The exponential function $f(x) = a^x$ is one-to-one, since if $f(x) = f(y)$, then $a^x = a^y$, which means that $a^{x-y} = 1$ and so $x - y = 0$. So a^x must have an inverse function.

Definition 8.20. The *logarithmic function with base* a, written $\log_a$, is the inverse function to a^x. It has domain $(0, +\infty)$, and its image is all real numbers. We often write ln for $\log_e$.

Thus if $a > 0$, we see that $\log_a(a^x) = x$ for every real x, and $a^{\log_a(x)} = x$ for every $x > 0$.

Example 8.21.

- $\log_3(9) = 2$.
- $\log_2(8) = 3$

- $\log_a(1) = 0$ for any $a > 0$.

Proposition 8.22. *If $a > 1$, then $\lim_{x\to+\infty} \log_a(x) = +\infty$ and $\lim_{x\to-^+} \log_a(x) = -\infty$.*

The logarithm also has a number of properties corresponding to the exponential laws:

Proposition 8.23.
- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
- $\log_a(x^r) = r \log_a(x)$ *for any real number r.*

Example 8.24.
- $\ln(a) + \frac{1}{2}\ln(b) = \ln(a) + \ln(b)^{1/2} = \ln(a\sqrt{b})$.
- Solve $e^{5-3s} = 10$. We have that $5 - 3x = \ln 10$ and so $x = \frac{5-\ln 10}{3}$.

Remark 8.25. These properties are actually historically why the logarithm was originally important. Before calculators, people doing difficult computational work had to work by hand. Adding five digit numbers is much, much easier than multiplying them. So engineers would take the log of the numbers, add them together, and then exponentiate. This was all done with the help of massive books called log tables that would tell you the logarithm of a given number. Slide rules are essentially a way of making the log tables portable; but they were superseded by pocket calculators.

There is one more important logarithmic formula, corresponding to the exponential law I left out:

Proposition 8.26 (change of base)**.** *For any positive number $a \neq 1$, we have $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$.*

Proof. $\exp(\log_a(x) \cdot \ln(a)) = a^{\log_a(x)} = x$, so $\log_a(x) \cdot \ln(a) = \ln(x)$. ☐

This allows us to convert logs in any base to logs in another base.

Example 8.27. What is $\log_2 10$? By the change of base formula, we have $\log_2(10) = \frac{\ln 10}{\ln 2}$. $\ln 10 \approx 2.3$ and $\ln 2 \approx .7$, so $\log_2 10 \approx 2.3/.7 \approx 23/7$.

8.3 Derivatives of exponentials and logs

Now we're ready to start computing derivatives. Recall that $e = \lim_{x\to 0}(1 + x)^{1/x}$.

Proposition 8.28. *The function $f(x) = \log_a(x)$ is differentiable, with derivative $f'(x) = \frac{1}{x}\log_a e$.*

Proof.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\log_a(x+h)-\log_a(x)}{h} \\
&= \lim_{h\to 0} \frac{\log_a((x+h)/x))}{h} \\
&= \lim_{h\to 0} \frac{\log_a(1+\frac{h}{x})}{h} \\
&= \frac{1}{x}\lim_{h\to 0} \frac{x}{h}\log_a(1+\frac{h}{x}) \\
&= \frac{1}{x}\lim_{h\to 0} \log_a\left((1+\frac{h}{x})^{x/h}\right) \\
&= \frac{1}{x}\log_a\left(\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h/x}}\right) \\
&= \frac{1}{x}\log_a(e)
\end{aligned}
$$

□

Corollary 8.29. *If* $f(x) = \log_a(x)$ *then* $f'(x) = \frac{1}{x \ln a}$.

Proof. By the change of base formula, $\log_a(e) = \frac{\ln(e)}{\ln(a)}$. □

Corollary 8.30. $\ln'(x) = \frac{1}{x}$.

Remark 8.31. An alternate path to discover the natural logarithm is to ask "what is the function whose derivative is $1/x$?" We will mention this line of thought briefly at the end of class.

Example 8.32.

- Let $f(x) = \ln(x^3+1)$. Then $f'(x) = \frac{1}{x^3+1} \cdot 3x^2$.
- Let $g(x) = \log_a(\cos(x))$. Then $g'(x) = \frac{1}{\cos(x)\ln(a)} \cdot (-\sin(x)) = -\tan(x)/\ln(a)$.
- If $h(x) = \ln(|x|)$ then $h'(x) = 1/x$ if $x > 0$ and $h'(x) = (-1/x)\cdot(-1) = 1/x$ if $x < 0$. So $h'(x) = \frac{1}{x}$.

We can sometimes use logarithms and implicit differentiation to make difficult differentiation problems easier, just as we use them to simplify difficult arithmetic problems.

Example 8.33 (Power Rule)**.** If r is a real number and $f(x) = x^r$, then

$$\begin{aligned} y &= x^r \\ \ln|y| &= r\ln|x| \\ \frac{1}{y}\frac{dy}{dx} &= r\frac{1}{x} \\ \frac{dy}{dx} &= r\frac{y}{x} = rx^{r-1}. \end{aligned}$$

And finally, we can use the logarithmic derivatives to figure out the derivative of exp.

Proposition 8.34. *If $f(x) = a^x$ for $a > 0$, then f is differentiable and $f'(x) = a^x \ln a$.*

Proof.

$$\begin{aligned} y &= a^x \\ \ln|y| &= x\ln|a| \\ \frac{1}{y}\frac{dy}{dx} &= \ln a \\ \frac{dy}{dx} &= y\ln a = a^x \ln a. \end{aligned}$$

□

Corollary 8.35. $\exp'(x) = \exp(x)$.

Example 8.36.

- If $f(x) = e^{\sin(x)}$ then $f'(x) = e^{\sin(x)} \cdot \cos(x)$.
- If $g(x) = 5^{x^2+1}$ then $g'(x) = \ln(5)5^{x^2+1} \cdot 2x$.

Poll Question 8.3.1. If $h(x) = x^x$ we have to be *very careful*—the obvious approaches don't actually work. But logarithmically:

$$\begin{aligned} y &= x^x \\ \ln|y| = x\ln|x| & \\ \frac{1}{y}\frac{dy}{dx} &= \ln|x| + \frac{x}{x} = \ln|x| + 1 \\ \frac{dy}{dx} &= x^x(\ln|x| + 1). \end{aligned}$$

So $h'(x) = (\ln|x| + 1)x^x$.

You can get the same result by writing $h(x) = e^{x\ln(x)}$, and thus $h'(x) = e^{x\ln(x)}(\ln(x)+1) = x^x(\ln(x) + 1)$.

Example 8.37. We wish to find the derivative of $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

$$\begin{aligned}
\ln y &= \frac{3}{4}\ln(x) + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{3}{4x} + \frac{2x}{2x^2+2} - \frac{3\cdot 5}{3x+2} \\
\frac{dy}{dx} &= y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right) \\
&= \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right).
\end{aligned}$$

8.4 Inverse Trigonometric Functions

We can invert some polynomials, and we can invert exponential functions. The other very common sort of function to work with is a trigonometric function, and we'd like to find inverses to these as well.

As a straightforward question, we cannot invert the trigonometric functions because they are all periodic, and thus not one-to-one. For instance, $\sin(0) = \sin(\pi) = \sin(2\pi) = \sin(n\pi)$ for any integer n.

However, sometimes a function is invertible if you restrict its domain enough, e.g. to between two critical points. In this section we make canonical domain choices for the trigonometric functions such that they are invertible.

Definition 8.38. If $-1 \leq x \leq 1$, we define $\arcsin(x) = \sin^{-1}(x) = y$ where $\sin(y) = x$ and $-\pi/2 \leq y \leq \pi/2$.

arcsin has a domain of $[-1, 1]$ and a range of $[-\pi/2, \pi/2]$.

Example 8.39. We can determine that $\arcsin(-\sqrt{3}/2) = -\pi/3$ since $\sin(-\pi/3) = -\sqrt{3}/2$. (Of course, $\sin(5\pi/3) = -\sqrt{3}/2$ as well, but $5\pi/3 > \pi/2$).

With more cleverness, we can calculate $\cos(\arcsin(1/3))$. Suppose $\theta = \arcsin(1/3)$. Then θ is the angle of a triangle with opposite side of lenght 1 and hypotenuse of length 3; using the Pythagorean theorem we determine that the other side has length $\sqrt{8} = 2\sqrt{2}$. Since $\cos(\theta)$ is the length of the adjacent side over the hypotenuse, we have $\cos(\arcsin(1/3)) = 2\sqrt{2}/3$.

We can make similar definitions for inverse cosine and inverse tangent functions. We do have to be careful about the precise domains and images.

Definition 8.40. If $-1 \leq x \leq 1$, we define $\arccos(x) = \cos^{-1}(x) = y$ where $\cos(y) = x$ and $0 \leq y \leq \pi$. This function has domain $[-1, 1]$ and range $[0, \pi]$.

If x is a real number, we define $\arctan(x) = \tan^{-1}(x) = y$ where $\tan(y) = x$ and $-\pi/2 < y < \pi/2$. This function has domain $(-\infty, +\infty)$ and image $(-\pi/2, \pi/2)$.

$\lim_{x \to +\infty} \arctan(x) = \pi/2$ and $\lim_{x \to -\infty} \arctan(x) = -\pi/2$.

sin and cos and tan are all differentiable functions, so by the Inverse Function Theorem, so are arcsin and arccos and arctan, at least most of the time.

Proposition 8.41.
- $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$
- $\arccos'(x) = \frac{-1}{\sqrt{1-x^2}}$
- $\arctan'(x) = \frac{1}{1+x^2}$.

Proof. There are two approaches to proving these facts. One involves trigonometric identities, and the other involves thinking about triangles. They both involve implicit differentiation.

Suppose $y = \arcsin(x)$. Then $\sin(y) = x$ and thus $\cos(y)\frac{dy}{dx} = 1$. Then we have $\frac{dy}{dx} = \frac{1}{\cos(y)}$.

From here, we can say two things. One is that $\cos(y) = \sqrt{1 - \sin^2(y)} = \sqrt{1 - x^2}$, using the trigonometric identity that $\cos^2(y) + \sin^2(y) = 1$ and being careful about sign choices.

I find it easier to think the following thing: if $y = \arcsin(x)$ then y is the angle of a triangle where the opposite side has length x and the hypotenuse has length 1. Then the other side has length $\sqrt{1-x^2}$, so $\cos(y) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$.

Note we got the same answer both ways, and they both involved basically the same facts; the identity $\sin^2(y) + \cos^2(y) = 1$ holds precisely because of the triangle argument. Either way you want to think of it is fine with me.

We can do the same with $\arccos(x)$. $\cos(y) = x$, so $\frac{dy}{dx} = \frac{-1}{\sin(y)} = -\frac{1}{\sqrt{1-x^2}}$.

arctan is slightly trickier. $\tan(y) = x$ so $\sec^2(y)\frac{dy}{dx} = 1$, and thus we have $\frac{dy}{dx} = \cos^2(y)$. Again, we can use the identity $1 + \tan^2(y) = \sec^2(y)$, but if we don't remember that we can see that y is the angle of a triangle with opposite side x and adjacent side 1, and hence hypotenuse $\sqrt{1+x^2}$. Then $\cos(y) = \frac{1}{\sqrt{1+x^2}}$ and so $\arctan'(x) = \cos^2(y) = \frac{1}{1+x^2}$. □

Example 8.42. What is $\arcsin'(.75)$? $\frac{1}{\sqrt{1-9/16}} = \frac{1}{\sqrt{7/16}}$.

What is $\arctan'(e^x)$? $\frac{1}{1+e^{2x}} \cdot e^x$.

What is $\arccos'(x^2 + 2x + 3)$? $\frac{1}{\sqrt{1-(x^2+2x+3)^2}} \cdot (2x + 2)$.

8.5 L'Hôpital's Rule

We often find ourselves wanting to evaluate limits of "indeterminate form": that is, the limit of a quotient whose numerator and denominator both approach 0 or both approach $\pm\infty$. In the past we've used various tricks to work out such limits, but today we develop a new and widely-applicable tool. This tool is especially useful for dealing with limits involving ln or exp.

Theorem 8.43 (L'Hôpital's Rule). *Suppose f and g are differentiable, and $g'(x) \neq 0$ near a, except possibly at a. Suppose either $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$. (In other words, the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form). Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right exists.

Remark 8.44. Note that L'Hôpital's Rule *only* applies to limits of indeterminate form.

Proof. We won't prove this fully, but we will prove it in the case where $f(a) = g(a) = 0$, $g'(a) \neq 0$, and f' and g' are continuous at a.

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} \\
&= \lim_{x\to a} \frac{(f(x) - f(a))(x-a)}{(g(x) - g(a))(x-a)} \\
&= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \\
&= \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.
\end{aligned}$$

□

Example 8.45.

$$\lim_{x\to 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3} = \lim_{x\to 3} \frac{2x - 4}{2x - 2} = \frac{2}{4} = \frac{1}{2}.$$

$$\lim_{x\to 0} \frac{1 - \cos(x)}{\sin(x)} = \lim_{x\to 0} \frac{\sin(x)}{\cos(x)} = \frac{0}{1} = 0.$$

$$\lim_{x\to 1} \frac{\ln x}{x - 1} = \lim_{x\to 0} \frac{1/x}{1} = 1.$$

Sometimes we have to apply L'Hôpital's rule more than once to get the results we want.

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan x - x}{x^3} &= \lim_{x\to 0} \frac{\sec^2(x) - 1}{3x^2} \\
&= \lim_{x\to 0} \frac{2\sec^2(x)\tan(x)}{6x} = \lim_{x\to 0} \frac{\tan x}{3x} \\
&= \lim_{x\to 0} \frac{\sec^2(x)}{3} = \frac{1}{3}.
\end{aligned}$$

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$$

We can also use L'Hôpital's rule to evaluate limits at infinity.

Example 8.46.

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{x^2 + 5x + 3}{x^2 + 7x - 2} &= \lim_{x\to\pm\infty} \frac{2x + 5}{2x + 7} \\
&= \lim_{x\to\pm\infty} \frac{2}{2} = 1. \\
\lim_{x\to+\infty} \frac{\ln(x)}{x} &= \lim_{x\to+\infty} \frac{1/x}{1} = 0. \\
\lim_{x\to+\infty} \frac{e^x}{x} &= \lim_{x\to+\infty} \frac{e^x}{1} = +\infty.
\end{aligned}$$

In fact, it's not too hard to see, using L'Hôpital's Rule, that $\lim_{x\to+\infty} \frac{e^x}{x^n} = +\infty$ and $\lim_{x\to+\infty} \frac{\ln(x)}{x^n} = 0$.

Remember that L'Hôpital's rule only applies if we start with an indeterminate form.

Example 8.47.

$$\begin{aligned}
\lim_{x\to\pi} \frac{\sin(x)}{1 - \cos(x)} &\neq \frac{\cos(x)}{\sin(x)} = \pm\infty \\
\lim_{x\to\pi} \frac{\sin(x)}{1 - \cos(x)} &= \frac{0}{1 - (-1)} = 0.
\end{aligned}$$

A more dangerous example:

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^3} = \lim_{x\to 0} \frac{e^x - 1}{3x^2} = \lim_{x\to 0} \frac{e^x}{6x}$$

You might think we should use L'Hôpital's rule again here; that would give $\lim_{x\to 0} \frac{e^x}{6} = 1/6$. But the top goes to 1 and the bottom goes to 0, so this is not an indeterminate form! The true limit is $\pm\infty$.

And sometimes L'Hôpital's rule doesn't alwasy work the way we'd like it to, just "because it doesn't."

Example 8.48.

$$\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\pm\infty} \frac{1}{\frac{x}{\sqrt{x^2+1}}} = \lim_{x\to\pm\infty} \frac{\sqrt{x^2+1}}{x}$$

But here if we're clever we can observe that if the limit exists, then

$$\left(\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}}\right)^2 = 1$$
$$\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}} = \pm 1.$$

We can often use L'Hôpital's rule to compute limits of other indeterminate forms with a bit of cleverness. Recall the "minor" indeterminate forms are $1^\infty, \infty - \infty, 0^0, 0^\infty, 0 \cdot \infty$. Products can obviously be rewritten as quotients, and sums or differences can often be combined into something by collecting common denominators. Exponents can be turned into ratios by means of logarithms.

Example 8.49.

$$\begin{aligned}
\lim_{x\to\pi/2} \sec(x) - \tan(x) &= \lim_{x\to\pi/2} \left(\frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)}\right) \\
&= \lim_{x\to\pi/2} \frac{1-\sin(x)}{\cos(x)} \\
&= \lim_{x\to\pi/2} \frac{-\cos(x)}{-\sin(x)} = \frac{0}{1} = 0. \\
\lim_{x\to 0} \cot(2x)\sin(6x) &= \lim_{x\to 0} \frac{\sin(6x)\cos(2x)}{\sin(2x)} = 1 \cdot \lim_{x\to 0} \frac{\sin(6x)}{\sin(2x)} \\
&= \lim_{x\to 0} \frac{6\cos(6x)}{2\cos(2x)} = 3. \\
\lim_{x\to 1} x^{1/(1-x)} &= \exp\left(\lim_{x\to 1} \frac{\ln x}{1-x}\right) \\
&= \exp\left(\lim_{x\to 1} \frac{1/x}{-1}\right) = \lim_{x\to 1} e^{-1/x} = 1/e.
\end{aligned}$$

8.6 [Bonus Material] Exponential Growth and Differential Equations

When we're doing mathematical modelling, it's often useful to write down an equation that relates a quantity of one thing to a rate of change—that is, a derivative—of something else. Often we have an unknown function, but we know something about its derivative An equation like this is called a *differential equation*. There are many techniques of math used to start with a differential equation and determine the original function. (See for instance Math 341 and Math 342 for Oxy's courses specifically on differential equations).

Here we will study a very common kind of differential equation called exponential growth. Exponential growth arises when the rate of change of a variable is proportional to the current size of that variable. This is often true of population dynamics, for instance; if 2% of people reproduce in a given year, then the rate of change of the population is 1/50 of the number of people in it. This also comes up in nuclear decay, economic growth, interest on bank accounts, and many other contexts.

Mathematically, we represent this situation with the following differential equation:

$$\frac{dy}{dt} = ky.$$

That is, the rate of change of y is equal to y times some constant which we customarily label k. k is sometimes called the "rate of growth" or "rate of decay".

It's not difficult to see that $y(t) = Ce^{kt}$ satisfies this differential equation for any constant C. (If nothing else, you could probably guess from the name "exponential growth"). It's more difficult—and requires tools of integral calculus—to show convincingly that these are the only functions that satisfy the differential equation of exponential growth.

Remark 8.50. The theory of solving differential equations is large, but there are some others you can probably solve already. How many different types of functions can you come up with that solve $y'' = ky$? This equation governs the motion of things like pendulums and weights on a spring.

Hint: in addition to exponential functions, you might also want to think about trigonometric functions.

Example 8.51 (Population Growth). Let $P(t)$ be the size of a population of animals or people or Tribbles at time t. In the absence of resource restrictions, the population will grow at a rate $\frac{dP}{dt} = kP(t)$, where k is the rate of growth. Then we must have $P(t) = Ce^{kt}$. What is C? Well, if we evaluate at 0, we see that $P(0) = Ce^{k0} = C$, so C is the level of the population at time $t = 0$.

The total population of the world was 3 billion people in 1960, and 4 billion in about 1975. Setting $t = 0$ to be 1960 and fitting this to our model, we have: $Ce^{k0} = 3$ and $Ce^{15k} = 4$. Thus we must have $C = 3$, and then $e^{15k} = 4/3$ implies that $15k = \ln(4/3)$ and so $k = \ln(4/3)/15$. If we want to estimate global population in 2020, this gives us

$$P(60) = 3 \cdot e^{60 \cdot \ln(4/3)/15} = 3 \cdot e^{4 \ln(4/3)} = 3 \cdot (4/3)^4 \approx 9.48.$$

(Actual estimates put it at 7.7 billion, because population growth has been leveling off).

Now let's use our model to estimate when the population will reach 12 billion. We want

$$12 = 3 \cdot e^{t\ln(4/3)/15} \tag{1}$$
$$4 = \left(\frac{4}{3}\right)^{t/15} \tag{2}$$
$$\log_{4/3} 4 = t/15 \tag{3}$$
$$15\frac{\ln(4)}{\ln(4/3)} = t \tag{4}$$
$$72.3 \approx t \tag{5}$$

So our model predicts that the world's population will reach 12 billion in about 2032.

Example 8.52 (Compound Interest)**.** Exponential growth often turns up in the context of economic growth and interest. Suppose you invest \$100 in a bank account paying 3% interest a year. Then after t years you will have $100 \cdot (1.03)^t$ dollars in the bank account. It's easy to compute how much money you'll have after t years. For instance, after three years you will have \$109 and after 20 years you will have \$180.

Often interest is "compounded" more often, meaning that you get some fraction of it every few months. Interest that is compounded quarterly—four times a year—pays you .75% of your current balance four times a year, so after t years you will have $100 \cdot (1.0075)^{4t}$ dollars. After three years you will still have \$109, and after 20 years you will have \$182. Note that your money has increased—slightly.

We can compound more often; in general, if your interest rate is r and you compound n times a year, then your total money after t years will be

$$M = M_0(1 + \frac{r}{n})^{nt},$$

where M_0 is the amount of money you started with.

In the real economy, transactions are constantly happening and the economy is (usually) constantly growing. Jacob Bernoulli asked what would happen if your interest *compounded continuously*—that is, what happens in the limit, as n goes to $+\infty$.

$$\begin{aligned} M(t) &= \lim_{n\to+\infty} M_0\left(1+\frac{r}{n}\right)^{nt} \\ &= M_0 \lim_{n\to+\infty}\left(\left(1+\frac{r}{n}\right)^{n/r}\right) \\ &= M_0\left(\lim_{n\to+\infty}\left(1+\frac{r}{n}\right)^{n/r}\right)^{rt} \\ &= M_0e^{rt}. \end{aligned}$$

And now you see (one reason) why e is considered a "natural" base for the exponential map.

With this, we can ask how long it will take for us to have \$200 if our interest is compounded continuously. We have

$$\begin{aligned} 200 &= 100e^{.03t} \\ 2 &= e^{.03t} \\ \ln(2) &= .03t \\ 23 &= t \end{aligned}$$

so it will take about 23 years to double our money.

Remark 8.53. We know that $\ln(2) = .693\cdots \approx .7$. This gives us the useful rule of thumb that if your interest rate is r, it will take about $70/r$ years to double your investment.

Example 8.54 (Radioactive Decay)**.** Radioactive substances decay randomly; each atom has a 50% chance of decaying over a given period of time. (This time is called the *half-life*). This means that at any given time, the decay rate of an amount S of a radioactive substance is proportional to the amount of subtance; that is, we can write $\frac{dS}{dt} = kS$. This gives us that $S = S_0 e^{kt}$ where S_0 is the amout of radioactive substance at time $t = 0$.

The half-life of Radium-226 is 1590 years. How much of a $100g$ mass of radium will be left after 1000 years?

We want to find the value of the constant k. We know that $S(0) = 100$ and that $S(1590) = 50$. This gives us that

$$\begin{aligned} 100 &= Ce^{k\cdot 0} & &\Rightarrow C = 100 \\ 50 &= Ce^{k\cdot 1590} & 50 &= 100e^{k\cdot 1590} \\ \frac{1}{2} &= e^{k\cdot 1590} & \ln(1/2) &= 1590k \\ \ln(1) - \ln(2) &= 1590k & \frac{-\ln(2)}{1590} &= k. \end{aligned}$$

This tells us that

$$S(t) = 100e^{\frac{-t\ln(2)}{1590}}.$$

Thus

$$S(1000) = 100e^{\frac{-1000\ln(2)}{1590}} \approx 100e^{-.436} \approx 64.7g.$$

So after 100 years we will have 64.7 g of radium-226 left. This should pass a sanity check, since it will take us 1590 years to get to 50.

Another question: When will we reach 30g?

We write

$$
\begin{aligned}
30 &= 100e^{\frac{-t\ln(2)}{1590}} \\
\ln(3/10) &= \frac{-t\ln(2)}{1590} \\
1590(\ln(3/10)) &= -t\ln(2) \\
-1590(\ln(3/10)/\ln(2)) &= t \\
-1590(-1.2/.7) &\approx t \\
2762 &\approx t
\end{aligned}
$$

So we will be down to 30g of radium-226 after about 2762 years. This again seems reasonable: it should take a bit less than 3180 years to reach 30g.