Math 114 Spring 2017
Calculus I HW 3 Solutions
Due Friday, February 10

1. Let
$$f(x) = \begin{cases} 1 & x < 2 \\ 2 & x = 2 \\ 3 & x > 2 \end{cases}$$
What is $f(2)$? Prove that $\lim_{x\to 2} f(x)$ does not exist.

   Solution: $f(2) = 2$ by definition of the function.

   Suppose $\lim_{x\to 2} f(x) = L$. Set $\epsilon = 1$; we can choose $\delta > 0$ so that when $0 < |x-2| < \delta$ then $|f(x) - L| < \epsilon = 1$.

   Let $x_1 = 2+\delta/2$, so that $|x_1 - 2| = \delta/2 < \delta$; then $f(x_1) = 3$, and thus $1 > |f(x_1) - L| = |3 - L|$.

   Let $x_2 = 2 - \delta/2$, so that $|x_2 - 2| = \delta/2 < \delta$; then $f(x_2) = 1$ and $1 > |f(x_2) - L| = |1 - L| = |L - 1|$. (We flip the sign inside the absolute value to make the next step easier, when we would like the Ls to cancel).

   Adding these two inequalities and applying the triangle inequality gives us
$$1 + 1 > |L - 1| + |3 - L| \geq |(L - 1) + (3 - L)| = 2,$$
but this is a contradiction, so the limit must not exist.

2. Let
$$j(x) = \begin{cases} x - 1 & x < 0 \\ x + 1 & x \geq 0 \end{cases}$$
Show that $\lim_{x\to 3} j(x) = 4$.

   Solution: Let $\epsilon > 0$ and fix $\delta \leq \underline{3, \epsilon}$. Then if $0 < |x - 3| < \delta$ we know that $x > 0$, and so we have
$$|j(x) - 4| = |x + 1 - 4| = |x - 3| < \delta = \epsilon.$$

3. ($\star$)For the same function j, show that $\lim_{x\to 0} j(x)$ does not exist.

   Solution: Suppose $\lim_{x\to 0} j(x) = L$. Fix $\epsilon = \underline{1}$ and suppose $\delta > 0$. Pick $x_1 = \delta/2$ and $x_2 = -\delta/2$.

Then since $0 < |x_1 - 0| = \delta/2 < \delta$, we have

$$\epsilon > |j(x_1) - L| = |\delta/2 + 1 - L|.$$

Similarly, since $0 < |x_1 - 0| = \delta/2 < \delta$, we have

$$\epsilon > |j(x_1) - L| = |-\delta/2 - 1 - L| = |L + 1 + \delta/2|.$$

Adding these equations gives

$$\begin{aligned} 2\epsilon &> |\delta/2 + 1 - L| + |L + 1 + \delta/2| \\ &\geq |\delta/2 + 1 - L + L + 1L\delta/2| = |\delta + 2| \\ &= 2 + \delta. \end{aligned}$$

Since $\epsilon = 1$ this gives us $2 > 2 + \delta$ and thus $0 > \delta$ which is a contradiction. So no such limit exists.

4. ($\star$) Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

   Solution: Suppose $\lim_{x\to 0} \frac{|x|}{x} = L$. Set $\epsilon = 1$; we can choose $\delta > 0$ so that when $0 < |x| < \delta$ then $\left|\frac{|x|}{x} - L\right| < \epsilon = 1$.

   Let $x_1 = \delta/2 < \delta$; then

$$1 = \epsilon > \left|\frac{|x_1|}{x_1} - L\right| = \left|\frac{|\delta/2|}{\delta/2} - L\right| = |1 - L|.$$

   Let $x_2 = -\delta/2$ so that $|x_2| = \delta/2 < \delta$; then

$$1 = \epsilon > \left|\frac{|x_2|}{x_2} - L\right| = \left|\frac{|-\delta/2|}{-\delta/2} - L\right| = |-1 - L| = |L + 1|.$$

   Then adding the two inequalities and using the triangle inequality gives us

$$2 = 1 + 1 > |L + 1| + |1 - L| \geq |(L + 1) + (1 - L)| = 2,$$

   but this is a contradiction, so the limit must not exist.

5. Let

$$g(x) = \begin{cases} 2x & x < 2 \\ 5x^2 - 7 & x \geq 2 \end{cases}$$

   Find (with proof) $\lim_{x\to 2^-} g(x)$.

   Solution: Let $\epsilon > 0$ and let $\delta = \epsilon/2$. Then if $2 - \delta < x < 2$, we observe that $2 - x < \delta$ and compute

$$\begin{aligned} |g(x) - 4| &= |2x - 4| = 2|x - 2| \\ &= 2(2 - x) && \text{because } x < 2 \\ &< 2\delta = \epsilon. \end{aligned}$$

6. Let

$$h(x) = \begin{cases} x & x < -1 \\ x^2 & x > -1 \end{cases}$$

Find (with proof) $\lim_{x\to -1^+} h(x)$.

Solution: Let $\epsilon > 0$ and set $\delta \leq \underline{2, \epsilon/2}$. Then if $-1 < x < -1+\delta$ (And thus $0 < x+1 < \delta$), we have

$$|h(x) - 1| = |x^2 - 1| = |x-1| \cdot |x+1| < |x-1|\delta.$$

But we know that $-2 < x - 1 < \delta - 2$ are all negative numbers since $\delta \leq 2$, so we have $2 > 1 - x > 2 - \delta$ and thus $|x - 1| < 2$. So we have

$$|h(x) - 1| < |x-1|\delta < 2\delta \leq \epsilon.$$

7. Let a and c be any constants. Prove that $\lim_{x\to a} c = c$.

Solution: Let $\epsilon > 0$ and let $\delta = 1$. Then if $0 < |x - a| < \delta$, we compute

$$|f(x) - c| = |c - c| = 0 < \epsilon.$$