Lab 7 Thursday March 23

Approximating Functions with Derivatives

In the first part of this lab we will look at how the derivative of a function approximates that function.

Last week, we drew secant lines, which are lines that intersect the graph of a function in (at least) two points; we may recall that by rearranging some information, we can write

$$f(x) = \frac{f(x) - f(a)}{x - a}(x - a) + f(a).$$

As x approaches a, this becomes closer to being a tangent line, and the slope term becomes closer to $f'(a)$. Thus we can get a decent approximation, if x and a are close, by replacing this difference quotient with the derivative:

$$f(x) \approx f'(a)(x - a) + f(a).$$

Another approach to this same idea is to think geometrically, about the tangent line. Last week in lab we drew tangent lines, and saw that the tangent line to a function looks very similar to that function close up. That is, the tangent line to f at a is the line that most resembles f near the point a. Thus if we write the equation to the tangent line

$$y = f'(a)(x - a) + f(a)$$

then we get the line which most closely approximates the function f; we can plug x values into this formula to find approximate calculations of $f(x)$, with the approximation improving if x is closer to a.

If we want to compute `f[b]` for some specific and awkward value `b`, we can:

(a) Find a value a which is close to b, such that `f[a]` is easy to compute.

e.g. if $f(x) = \sqrt{x}$ and $b = 5$ we may pick $a = 4$.

(b) Calculuate the derivative at the point a: `f'[a]`.

If $f(x) = \sqrt{x}$ and $b = a$ then $f'(a) = \frac{1}{2\sqrt{a}} = \frac{1}{4}$.

(c) Find an equation $y = m(x - a) + f(a)$ for the line tangent to f at a. (Check your work by plotting f and your line on the same plot in Mathematica). What is the y value given by this line when the x-coordinate is b?

We know the slope of the tangent line at a is $\frac{1}{4}$, and $f(a) = 2$, so our line is $y = \frac{1}{4}(x - 4) + 2$. We can plot this in Mathematica with

`Plot[{Sqrt[x], (1/4) (x-4) + 2},{x,2,6}]` or

`Plot[{Sqrt[x], Sqrt'[4](x-4) + Sqrt[4]},{x,2,6}]`

When we plug in 5 (our b value) for x we get $y = \frac{1}{4}(5 - 4) + 2 = \frac{1}{4} + 2 = \frac{9}{4}$. Thus we guess $\sqrt{5} \approx \frac{9}{4}$.

(d) As an alternative way of thinking about this problem, calculate $(b - a)f'(a) + f(a)$. (You can think about this as starting from the value $f(a)$ and then adding a "correcting factor" based on the distance from a to b and the rate of change). Did you get the same thing as in part (c)? Why?

We calculate $(5-4)\frac{1}{4}+2 = \frac{9}{4}$, which is the same as before. We can also plug this into Mathematica with `(5-4) * Sqrt'[4] + Sqrt[4]` This is in fact the same formula, described differently, so it is unsurprising we get the same answer.

(e) Check your answers from (c) and (d) by evaluating `f[a]` (or more likely `N[f[a]]`) in Mathematica. How accurate were you?

Mathematica tells us that `N[Sqrt[5]]` is 2.23607, which is pretty close to 2.25. If we want to do better, in a few weeks we will cover a tool called "Newton's Method" that allows us to refine our answers and get closer approximations. Alternatively, the tool of "Taylor series" which is covered at the end of Math 120 (Calculus 2) gives another improved method of approximations, which uses the both the first and second (and potentially higher) derivatives.

Now answer the following questions. Before you do any computations, think about what you should use for f, a, b.

1. Approximate $(2.1)^5$.

   Solution: We take $f(x) = x^5$ and $a = 2$. Then $f'(x) = 5x^4$, so we have $f(2) = 32, f'(2) = 80$, and
   $$f(2.1) \approx 80(2.1 - 2) + 32 = 40.$$
   The exact answer is 40.841.

2. How about $(4.5)^3$?

   Solution: We take $f(x) = x^3$ and $a = 4$. Then $f(a) = 64$, and $f'(x) = 3x^2$ so $f'(a) = 48$, and
   $$f(4.5) \approx 48(4.5 - 4) + 64 = 24 + 64 = 88.$$
   Alternatively, we can take $f(x) = x^3$ and $a = 5$. Then $f(a) = 125, f'(x) = 3x^2, f'(a) = 75$, and
   $$f(4.5) \approx 75(4.5 - 5) + 125 = 125 - 37.5 = 87.5.$$
   The exact answer is 91.125.

3. Approximate `Sin[.05]` and `Cos[.05]` Note that this is .05 and not .5. Approximate $\sin(b)$ and $\cos(b)$ for b "small". (This is the revenge of the Small Angle Approximation).

   Solution: We take $a = 0$. Then since $\sin'(x) = \cos(x)$ and so $\sin'(0) = \cos(0) = 1$, we have
   $$\sin(.05) \approx 1(.05 - 0) + 0 = .05.$$
   This is basically the small angle approximation: for small x, we have $\sin(x) \approx x$. (The true answer is about .04998).

   Similarly, $\cos'(x) = -\sin(x)$ so $\cos'(0) = 0$. Then
   $$\cos(.05) \approx 0(.05 - 0) + 1 = 1.$$
   For small x, we have $\cos(x) = 1$. (The true answer is about .9986).

4. Now try `Sin[3/4]` and `Cos[3/4]` (Think about your choices for a here; you can do much better than 1) .

   Solution: Cheating only a little bit, we pick $a = \pi/4$, since $\pi \approx 3$. Then
   $$\sin(3/4) \approx \cos(\pi/4)(3/4 - \pi/4) + \sin(\pi/4) = \frac{\sqrt{2}}{8}(3 - \pi) + \frac{\sqrt{2}}{2} \approx \sqrt{2}/2(1 - .035)$$
   $$\cos(3/4) \approx -\sin(\pi/4)(3/4 - \pi/4) + \cos(\pi/4) = -\frac{\sqrt{2}}{8}(3 - \pi) + \frac{\sqrt{2}}{2} \approx \sqrt{2}/2(1 + .035).$$

5. Approximate `CubeRoot[28]` and `82^(1/4)`.

   Solution: We take $a = 27$ and $a = 81$ respectively.

$$\sqrt[3]{28} \approx \frac{1}{3}(27)^{-2/3}(28-27)+3 = \frac{1}{27}+3 \approx 3.03704$$
$$\sqrt[4]{82} \approx \frac{1}{4}(81)^{-3/4}(82-81)+3 = \frac{1}{108}+3 \approx 3.00926.$$

   The true answers are approximately 3.03659 and 3.00922 respectively.

6. Use the same method to find `CubeRoot[64.1]`.

   Solution: We take $a = 64, f(x) = \sqrt[3]{x}, f(a) = 4$. Then

$$\sqrt[3]{64.1} \approx \frac{1}{3}(64)^{-2/3}(64.1-64)+4 = \frac{1}{48}+4 \approx 4.02083.$$

   The true answer is about 4.00208.

7. Approximate $(1.01)^{10}$. Approximate $(1.01)^{\alpha}$ where $\alpha \neq 0$ is some constant (your answer will have an α in it). Approximate $(1+\epsilon)^{\alpha}$ where ϵ is a "small" constant and $\alpha \neq 0$ is a constant. (This rule is the binomial approximation and is often useful in physics).

   Solution: For the first part, our function is $f(x) = x^{10}$ and our $a = 1$. So $f(a) = 1$ and $f'(a) = 10a^9 = 10$. Then we have

$$f(1.01) \approx 10(1.01-1)+1 = 1.1.$$

   The true answer is about 1.10462.

   For the second part, we have $f(x) = x^{\alpha}$, so $f'(x) = \alpha x^{\alpha-1}$. We again have $f(1) = 1$ and $f'(1) = \alpha(1)^{\alpha-1} = \alpha$, so

$$f(1.01) \approx \alpha(1.01-1)+1 = 1+\alpha/100.$$

   For the third part, we still take $f(x) = x^{\alpha}$ and $a = 1$. But we compute

$$f(1+\epsilon) \approx \alpha(1+\epsilon-1)+1 = 1+\alpha\epsilon.$$

   This formula is used constantly in physics and other applications.

8. If you take $a = 0$ and $f(x) = x^{10}$, approximate $f(2)$. What happens and why? What if you instead approximate with $a = 1$?

   Solution: We have $f'(x) = 10x^9$, so we have $f'(0) = 0$, and thus

$$f(2) \approx 0(2-0)+0 = 0.$$

   If we take $a = 1$, we have

$$f(2) \approx 10(2-1)+1 = 11.$$

   The true answer is 1024, which is far away from both of those. In essence, the derivative is changing so quickly that the tangent line approximation is not very good over those distances.