Lab 9   Thursday April 6

Newton's Method

In this lab we will study the use of Newton's Method to find roots of functions. This is a further application of the method of linear approximation we studied in lab 7.

Suppose we have a function f and we want to find an approximate value for x where $f(x) = 0$ (that is, find an approximate root of f). If we know the value of f and of f' at a point x_1, then recall that by linear approximation we estimate that $f(x_2) = f(x_1) + f'(x_1)(x_2 - x_1)$. Since we want $f(x_2) = 0$, we set $f(x_2) = 0$ and solve this equation for x_2, and get

$$x_2 = x_1 - (f(x_1)/f'(x_1)).$$

In many conditions, we will get the result that x_2 is closer to being a root of f than x_1 is.

We can repeat this process to find x_3, x_4, etc., and ideally each will be a better estimate than the previous estimate was. A good rule of thumb for when to stop: if you want five decimal places of accuracy, you can stop when the nth step and the $n + 1$st step agree to five decimal places.

This method does have limitations. First, we have to start with a guess x_1 for our root x. Second, if $f'(x_1)$ is very close to zero, Newton's method will work poorly if it works at all, and we might have to pick a better guess. Think about what's happening when you use Newton's method. But it can be very useful for finding approximate solutions to equations.

Note that among other things, we can use this to calculate inverse functions; if I can compute $f(x)$ easily and wish to compute $f^{-1}(a)$, I simply search for approximations to $f(x) = a$.

Useful Mathematica note: `N[x,n]` will give you a numerical approximation of `x` with `n` digits of precision.

Longer useful Mathematica note: recall that we can evaluate the *same* line of code repeatedly, and we can use this fact to save a great deal of typing here. If we have defined a function `f` and a starting point `a`, we can run the following line of code repeatedly:
`a = a - f[a]/f'[a]`   and it will repeat the process over and over again; we can just keep hitting shift+enter until we have the precision we wish.

1. (a) Starting with $x_1 = 1$, use `f[x_] : = x^4 - 2` to estimate $\sqrt[4]{2}$ to four decimal places. (Notice the extra work we've done here: a fourth root of 2 is a solution to the equation $x^4 = 2$ and thus a root of $x^4 - 2$. In general we need to turn every problem into the question of finding a *root* of a function).

   Do two iterations by hand, and then shift to Mathematica. It might be helpful to start Mathematica off with `x=1.` (note the decimal point!) to force the software to give you decimal answers, which are more readable.

   (b) Plot the tangent line corresponding to the first two steps you did in part (a).

   (c) Repeat part (a) from the beginning starting from $x_1 = -1$, and again starting from $x_1 = 0$.

   (d) In Mathematica, run the commands `FindRoot[x^4 ==2, {x,1}]`, `FindRoot[x^4 == 2, {x,-1}]`, and `FindRoot[x^4 == 2, {x, 0}]`.

2. (a) Plot a graph of both `Cos[x]` and `x` with the command `Plot[{Cos[x],x},{x,-2Pi, 2Pi}]`. About where does it look ike the two functions intersect?

   (b) Using your guess from part (a) as a starting point, use Newton's method to estimate a solution to $\cos(x) = x$ that is correct to six decimal places.

   (c) Run the command `FindRoot[Cos[x]==x, {x,a}]`, where a is your guess from part (a).

3. (a) Starting with $x_1 = 1$, estimate the root to `g[x_]=x^3-x-1` to four decimal places.

   (b) Do the same, starting with $x_1 = .6$.

   (c) Do the same, starting with $x_1 = .57$.

   (d) Plot g from -2 to 2. Why were (a), (b), and (c) so different? Try plotting some tangent lines.

4. (a) Starting with $x_1 = 1$, use three iterations of Newton's method to find a solution to `CubeRoot[x] == 0`. What happens?

   (b) Plot a graph of `CubeRoot[x]`. Graphically, why did you get the result you did in part (a)? Plot the tangent lines that correspond to the approximations you calculated.

5. Let $f(x) = x^5 + x^3 + x$.

   (a) Use Newton's method to approximate $f^{-1}(2)$–i.e., to approximate a solution for $f(x) = 2$.

   (b) Use the Implicit Function Theorem to approximate $(f^{-1})'(2)$.

6. Let $g(x) = \sqrt{1 + x + x^2 + x^3}$.

   (a) Use Newton's Method to approximate $g^{-1}(3)$.

   (b) Use the Implicit Function theorem to approximate $(g^{-1})'(3)$.

7. (a) Approximate $\sqrt[5]{20}$ to eight decimal places.

   (b) Find four real roots of $x^6 - x^5 - 6x^4 - x^2 + x + 10$ to eight places.

   (c) Show that $x^4 - 3x^3 + 5x^2 - 6$ has a root in $(1,2)$ (hint: IVT), and approximate it to six decimal places.

   (d) Approximate $\log 3$.

Bonus: Does $3^{x^2} - 9^{4x} - 6$ have a root? (How many does it have?) Can you estimate them?

Go check out this Quora discussion and see a few different ways to approach this. (Especially read the post by Alon Amit. The solution by Prahar Mitra uses Taylor series, which I have alluded to several times but are not covered until Calculus II).

`https://www.quora.com/How-do-you-solve-3-x-2-9-4x-+6/answer/Richard-Muller-3`