Lab 9 Solutions Thursday November 10

Newton's Method

1. (a) Starting with $x_1 = 1$, use `f[x_] : = x^4 - 2` to estimate $\sqrt[4]{2}$ to four decimal places.

   We have $f'(x) = 4x^3$, so compute

$$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{4} = 5/4$$
$$x_3 = 5/4 - \frac{f(5/4)}{f'(5/4)} = \frac{5}{4} - \frac{(625/256) - (512/256)}{125/16} = \frac{5}{4} - \frac{113/256}{125/16} = \frac{5}{4} - \frac{113}{125 \cdot 16} = \frac{2387}{2000}$$

   Switching to a computer,

$$x_4 = \frac{2387}{2000} - \frac{f(2387/2000)}{f'(2387/2000)} \approx 1.18923$$
$$x_5 = 1.18923 - \frac{f(1.18923)}{f'(1.18923)} \approx 1.18921.$$

   (b) Plot the tangent line corresponding to each step you executed in part (a).

   (c) Repeat part (a) starting with $x_1 = -1$, and again with $x_1 = 0$.

   You should get -1.18921 in the first case, and in the second case you get a divide-by-zero error (literally–a "singular Jacobian" means a derivative equal to zero).

   (d) In Mathematica, run the commands `FindRoot[x^4 ==2,  {x,1}]`, `FindRoot[x^4 == 2, {x,-1}]`, and `FindRoot[x^4 == 2, {x, 0}]`.

2. (a) Plot a graph of both `Cos[x]` and `x` with the command `Plot[{Cos[x],x},{x,-2Pi, 2Pi}]`. About where does it look ike the two functions intersect?

   (b) Using your guess from part (a) as a starting point, use Newton's method to estimate a solution to $\cos(x) = x$ that is correct to six decimal places.

   Note that we are not asking for a root of $\cos(x)$! We want to know when $\cos(x) = x$, that is, when $\cos(x) - x = 0$. So we want a root of $f(x) = \cos(x) - x$.

   Starting with a guess of $x_1 = \pi/4$ (other guesses, such as 1, are also reasonable), we get

$$x_2 = \pi/4 - \frac{f(\pi/4)}{f'(\pi/4)} = \pi/4 - \frac{\sqrt{2}/2 - \pi/4}{-\sqrt{2}/2 - 1} \approx .739536$$
$$x_3 = .739536 - \frac{f(.739536)}{f'(.739536)} = .739085$$

   (c) Run the command `FindRoot[Cos[x]==x, {x,a}]`, where a is your guess from part (a).

3. (a) Starting with $x_1 = 1$, estimate the root to `g[x_]=x^3-x-1` to four decimal places.
   We have $g'(x) = 3x^2 - 1$, so compute

$$x_2 = 1 - \frac{-1}{2} = \frac{3}{2}$$
$$x_3 = \frac{3}{2} - \frac{27/8 - 3/2 - 1}{27/4 - 1} \approx 1.34783$$
$$x_4 = 1.34783 - \frac{g(1.34783)}{g'(1.34783)} \approx 1.3252$$
$$x_5 \approx 1.32472$$

   (b) Do the same, starting with $x_1 = .6$.
   This should take much longer

   (c) Do the same, starting with $x_1 = .57$.
   This should take nearly a hundred iterations if it converges at all.

   (d) Plot g from -2 to 2. Why were (a), (b), and (c) so different? Try plotting some tangent lines.
   There is a local minimum at about .577. .6 is close to that but on the same side as the (only) root, so it converges but slowly. .57 is on the "wrong" side and gets "trapped" there for a long time.

4. (a) Starting with $x_1 = 1$, use three iterations of Newton's method to find a solution to `CubeRoot[x] == 0`. What happens?
   Starting with 1 we have

$$x_2 = 1 - \frac{\sqrt[3]{1}}{(1/3)1^{-2/3}} = 1 - 3 = -2$$
$$x_3 = -2 - \frac{\sqrt[3]{-2}}{1/3(-2)^{-2/3}} = -2 + 6 = 4$$
$$x_4 = -8$$

   and so on. The method never converges, and in fact each new iteration gets us further from the root (which is zero) than the previous one did.

   (b) Plot a graph of `CubeRoot[x]`. Graphically, why did you get the result you did in part (a)? Plot the tangent lines that correspond to the approximations you calculated.
   There is a vertical tangent line at the root.

5. Let $f(x) = x^5 + x^3 + x$.

   (a) Use Newton's method to approximate $f^{-1}(2)$–i.e., to approximate a solution for $f(x) = 2$.

We want a solution to $f(x) = 2$, so set $f_2(x) = x^5 + x^3 + x - 2$, and then we're looking for a root of f_2. We have $f_2'(x) = 5x^4 + 3x^2 + 1$. Take $x_1 = 1$ since we know that $f(1) = 3$ is close to 2; then we have

$$\begin{aligned} x_2 &= 1 - \frac{f_2(1)}{f_2'(1)} = 1 - \frac{1}{9} = \frac{8}{9} \\ x_3 &= \frac{8}{9} - \frac{f_2(8/9)}{f_2'(8/9)} = \frac{8}{9} - \frac{(8/9)^5 + (8/9)^3 + 8/9 - 2}{5(8/9)^4 + 3(8/9)^2 + 1} \approx .866376 \\ x_4 &= .865583 \end{aligned}$$

(b) Use The Implicit Function Theorem to approximate $(f^{-1})'(2)$.
We know from part (a) that $f^{-1}(2) \approx .865583$. So

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1})(2)} \approx \frac{1}{f'(.865583)} \approx \frac{1}{6.05446} \approx .165168.$$

6. Let $g(x) = \sqrt{1 + x + x^2 + x^3}$.

(a) Use Newton's Method to approximate $g^{-1}(3)$.
Again, we want a root of $g(x) - 3$ so let $g_1(x) = \sqrt{1 + x + x^2 + x^3} - 3$. Then $g_1'(x) = \frac{1+2x+3x^2}{2\sqrt{1+x+x^2+x^3}}$.
We can take $x_1 = 0$. Then

$$\begin{aligned} x_2 &= 0 - \frac{g_1(0)}{g_1'(0)} = 0 - \frac{-2}{1/2} = 4 \\ x_3 &= 4 - \frac{g_1(4)}{g_1'(4)} = 4 - \frac{\sqrt{85} - 3}{57/2\sqrt{85}} = 4 - \frac{2\sqrt{85}}{57}\left(\sqrt{85} - 3\right) \approx 1.988 \\ x_4 &= x_3 - \frac{g_1(x_3)}{g'(x_3)} \approx 1.60. \end{aligned}$$

The true answer is approximately 1.578.

(b) Use the Implicit Function theorem to approximate $(g^{-1})'(1)$.
We have $g^{-1}(3) \approx 1.6$ from part (c), and thus

$$(g^{-1})'(1) = \frac{1}{g'(g^{-1}(1))}\frac{1}{g'(1.6)} \approx \frac{1}{1.95} \approx .51.$$

7. (a) Approximate $\sqrt[5]{20}$ to eight decimal places.

(b) Find four real roots of $x^6 - x^5 - 6x^4 - x^2 + x + 10$ to eight places.

(c) Show that $x^4 - 3x^3 + 5x^2 - 6$ has a root in (1,2) (hint: IVT), and approximate it to six decimal places.

(d) Approximate $\log 3$.