Problem 1. (a) Use the definition of limit to prove that $\lim_{x\to 2} \frac{1}{x+3} = \frac{1}{5}$.

Solution: Let $\epsilon > 0$ and set $\delta \leq \underline{20\epsilon, 1}$. Then if $|x-2| < \delta$ we have

$$\begin{aligned}\left|\frac{1}{x+3} - \frac{1}{5}\right| = \left|\frac{2-x}{5(x+3)}\right| &= \frac{|x-2|}{5|x-2+5|} \\ &= \frac{|x-2|}{5|5-(2-x)|} \leq \frac{|x-2|}{5(5-|x-2|)} \\ &< \frac{\delta}{5(5-\delta)} \leq \frac{20\epsilon}{20} = \epsilon.\end{aligned}$$

(b) Use the definition of limit to prove that $\lim_{x\to 1} \frac{1}{(x-1)^2} = +\infty$.

Solution: Let $N > 0$ and set $\delta = \underline{1/\sqrt{N}}$. Then if $|x-1| < \delta$ we have

$$\frac{1}{(x-1)^2} = \frac{1}{|x-1|^2} > \frac{1}{\delta^2} = \frac{1}{1/N} = N.$$

Problem 2. (a) Use the Squeeze Theorem to show that $\lim_{x\to 5}(x-5)\sin\left(\frac{x^2+1}{x-5}\right) = 0$.

Solution: We have

$$\begin{aligned}-1 \leq \sin\left(\frac{x^2+1}{x-5}\right) &\leq 1 \\ -|x-5| \leq (x-5)\sin\left(\frac{x^2+1}{x-5}\right) &\leq |x-5|.\end{aligned}$$

We see that $\lim_{x\to 5} -|x-5| = \lim_{x\to 5} |x-5| = 0$, so by the squeeze theorem we know that

$$\lim_{x\to 5}(x-5)\sin\left(\frac{x^2+1}{x-5}\right) = 0.$$

(b) Compute $\lim\limits_{x\to 25} \dfrac{\sqrt{x}-5}{x-25}$

Solution:

$$\lim_{x\to 25} \frac{\sqrt{x}-5}{x-25} = \lim_{x\to 25} \frac{x-25}{(x-25)(\sqrt{x}+5)} = \lim_{x\to 25} \frac{1}{\sqrt{x}+5} = \frac{1}{10}.$$

(c) Compute $\lim\limits_{x\to 5} \dfrac{e^{x-5}-1}{x-5}$

Solution: We have that $\lim_{x\to 5} e^{x-5} - 1 = \lim_{x\to 5} x - 5 = 0$, so we can use L'Hôpital's rule.

$$\lim_{x\to 5} \frac{e^{x-5}-1}{x-5} =^{\text{L'H}} \lim_{x\to 5} \frac{e^{x-5}}{1} = 1.$$

Problem 3. (a) **Directly from the definition**, compute $f'(1)$ where $f(x) = \sqrt{x+3}$.

Solution:

$$\begin{aligned}f'(1) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{4+h}-\sqrt{4}}{h} \\ &= \lim_{h\to 0} \frac{(4+h)-4}{h(\sqrt{4+h}+\sqrt{4})} = \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{4}.\end{aligned}$$

(b) Compute $g'(x)$ where $g(x) = \ln \left| \frac{e^{\arctan(x^2)} - 5}{\sqrt[4]{x^2+1}} \right|$.

Solution:

$$g'(x) = \frac{1}{\frac{e^{\arctan(x^2)}-5}{\sqrt[4]{x^2+1}}} \cdot \frac{(e^{\arctan(x^2)}\frac{2x}{1+x^4})\sqrt[4]{x^2+1} - \frac{1}{4}(x^2+1)^{-3/4}2x(e^{\arctan(x^2)} - 5)}{\sqrt[2]{x^2+1}}$$

(c) Find a tangent line to the function $f(x) = \frac{e^x}{x}$ at the point given by $x = 2$.

Solution:

$$f'(x) = \frac{e^x \cdot x - e^x}{x^2},$$

so $f'(2) = \frac{2e^2 - e^2}{4} = \frac{1}{4}e^2$. Thus the tangent line has equation

$$y = \frac{1}{4}e^2(x-2) + \frac{1}{2}e^2.$$

Problem 4. (a) **Directly from the definition**, compute $f'(x)$ where $f(x) = \frac{1}{x-7}$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h-7} - \frac{1}{x-7}}{h} \\ &= \lim_{h\to 0} \frac{(x-7) - (x+h-7)}{(x-7)(x+h-7)h} = \lim_{h\to 0} \frac{-1}{(x-7)(x+h-7)} = \frac{-1}{(x-7)^2}. \end{aligned}$$

(b) Write a tangent line to the curve $y^2 = x^{x\cos(x)}$ at the point $(\pi/2, -1)$.

Solution: Implicit differentiation gives us

$$\begin{aligned} 2\ln(y) &= x\cos(x)\ln(x) \\ \frac{2y'}{y} &= \cos(x)\ln(x) - x\sin(x)\ln(x) + \cos(x) \\ y' &= \frac{1}{2}\left(\cos(x)\ln(x) - x\sin(x)\ln(x) + \cos(x)\right) y. \end{aligned}$$

When $x = \pi/2, y = -1$, this gives us

$$\begin{aligned} y' &= \frac{1}{2}\left(0\ln(\pi/2) - \pi/2 \cdot 1 \cdot \ln(\pi/2) + 0\right)(-1) = \frac{1}{2}\left(\pi/2\ln(\pi/2)\right) \\ &= \frac{\pi(\ln(\pi) - \ln(2))}{4} \end{aligned}$$

and thus the tangent line has equation

$$y = \frac{\pi(\ln(\pi) - \ln(2))}{4}(x - \pi/2) - 1.$$

(c) Find y' if $e^y + \ln(y) = x^2 + 1$.

Solution:

$$\begin{aligned} e^y \cdot y' + \frac{y'}{y} &= 2x \\ y'(e^y + \frac{1}{y}) &= 2x \\ y' &= \frac{2x}{e^y + \frac{1}{y}}. \end{aligned}$$

Problem 5. (a) A cone with height h and base radius r has volume $\frac{1}{3}\pi r^2 h$. Suppose we have an inverted conical water tank, of height 4m and radius 6m. Water is leaking out of a small hole at the bottom of the tank. If the current water level is 2m and the water level is dropping at $\frac{1}{9\pi}$ meters per minute, what volume of water leaks out every minute?

Solution: We have $V = \frac{1}{3}\pi r^2 h$ and $r = 3h/2$, and thus

$$\begin{aligned} V &= \frac{1}{3}\pi(\frac{3h}{2})^2 h = \frac{3}{4}\pi h^3 \\ V' &= \frac{9}{4}\pi h^2 h' \\ V' &= \frac{9}{4}\pi(2)^2 \frac{-1}{9\pi} = 1 \end{aligned}$$

So one cubic meter of water is leaking out every minute.

(b) Use two iterations of Newton's method, starting at 0, to estimate the root of $e^x - 3x$.

Solution: Set $f(x) = e^x - 3x$, and $x_1 = 0$. We have $f'(x) = e^x - 3$.

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{1-0}{1-3} = \frac{1}{2} \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{1}{2} - \frac{\sqrt{e} - \frac{3}{2}}{\sqrt{e} - 3} = \frac{\sqrt{e} - 3}{2(\sqrt{e} - 3)} - \frac{2\sqrt{e} - 3}{2(\sqrt{e} - 3)} = \frac{\sqrt{e}}{6 - 2\sqrt{e}}. \end{aligned}$$

(c) Let $g'(x) = g(x) + 3x$, and $g(2) = 4$. Use two steps of Euler's method to estimate $g(4)$. Is this an overestimate or an underestimate?

Solution:

$$\begin{aligned} g(3) &\approx g'(2)(3-2) + g(2) = 10(1) + 4 = 14 \\ g(4) &\approx g'(3)(4-3) + g(3) = 23(1) + 14 = 37. \end{aligned}$$

This is a wild underestimate because the derivative is increasing so rapidly.

Problem 6. (a) A radioactive substance begins decaying from 100g of material. When it reaches 10g, it is decaying at rate of 1g per year. After how many years does this occur?

Solution: If $S(t)$ is the amount of substance in year t, then we have $S(t) = Ce^{rt}$, and thus $S(0) = 100 = C$. We know that $S'(t) = rCe^{rt} = rS(t)$, so when $S(t) = 10$ we have $-1 = r10$ and thus $r = -1/10$. This gives us $S(t) = 100e^{-t/10}$. Now we can solve $10 = 100e^{-t/10}$, which implies $10^{-1} = e^{-t/10}$ and thus $-\ln(10) = -t/10$. Thus $t = 10\ln(10) \approx 23$ years.

(b) Suppose that a company that produces and sells x units of a product makes a revenue of $R(x) = 100x - x^2/20$ and has costs given by $C(x) = 700 + 40x - x^2/100$. What is the maximum profit that can be made (where profit is revenues minus costs)?

Solution: We have our profit function given by $P(x) = -700 + 60x - x^2/25$. Then $P'(x) = 60 - 2x/25$ has a critical point when $x = 750$. We see that $P''(x) = -2/25 < 0$ so this is a local maximum. We can see it is a global maximum in one of two ways:

First, we see that $P'(x) > 0$ when $x < 750$, and $P'(x) < 0$ when $x > 750$. Thus if we go either to the left or the right, the function will be lower than it is at 750, so the value at 750 is a local maximum.

Alternatively, we could note that $P(x) > 0$ only on some closed interval. We see that both endpoints have the value 0, and the only critical point is 750 with value > 0, so this must be the global maximum by the Extreme Value Theorem.

To complete the answer, we must know what the profit made with 750 units is. We have

$$P(750) = -700 + 60 \cdot 750 - 150^2 = -700 + 45000 - 22500 = 21800.$$

Thus the maximum possible profit is \$21800.

(c) Ten miles from home you remember that you left the water running, which is costing you 90 cents an hour. Driving home at speed s miles per hour costs you $4(s/10)$ cents per mile. At what speed should you drive to minimize the total cost of gas and water?

Solution: The water will be running for $10/s$ hours and thus the total cost of water will be $900/s$ cents. The cost of driving will be $10 \cdot 4(s/10) = 4s$ cents. Thus our total cost is $C(s) = 4s + 900/s$, and we want to minimize this.

We have $C'(s) = 4 - 900/s^2$. This has critical points at $s = 0$ and when $4s^2 = 900$ and thus $s^2 = 225$ and $s = \pm 15$. Clearly we must have $s > 0$ for physical reasons, so the only relevant critical point is $s = 15$.

Checking the second derivative we have $C''(s) = 1800/s^3$ and thus $C''(15) = 8/15 > 0$ and thus $s = 15$ is a local minimum. In fact s is the global minimum for positive values; we can see this since $C'(s) < 0$ when $0 < s < 15$ and $C'(s) > 0$ when $s > 15$. Thus you should drive at 15 miles per hour.

Problem 7. (a) Find the absolute extrema of $f(x) = 3x^4 - 20x^3 + 24x^2 + 7$ on $[0, 5]$.

Solution: f is a continuous function on a closed ineterval, so it must have an absolute maximum and an absolute minimum. $f'(x) = 12x^3 - 60x^2 + 48x = 12x(x^2 - 5x + 4) = 12x(x - 4)(x - 1)$ is defined everywhere and has roots at $0, 1, 4$. The endpoints are $0, 5$, so we need to evaluate f at $0, 1, 4, 5$.

$$\begin{aligned}
f(0) &= 7 \\
f(1) &= 14 \\
f(4) &= 3(4^4) - 5(4^4) + \frac{3}{2}(4^4) + 7 = \frac{-1}{2}4^4 + 7 = 7 - 128 = -121 \\
f(5) &= 3 \cdot 5^4 - 4 \cdot 5^4 + 5^4 - 5^2 + 7 = 7 - 25 = -18.
\end{aligned}$$

So the absolute maximum is 14 at 1, and the absolute minimum is -121 at 4.

(b) Find all the critical points of $f(x) = \dfrac{\ln(x)}{x^2 - 3x - 2}$

Solution: The function is undefined for $x \leq 0$.

$$\begin{aligned}
f'(x) &= \frac{\frac{1}{x}(x^2 - 3x - 2) - (2x - 3)\ln(x)}{(x^2 - 3x - 2)^2} = \frac{x^2 - 3x - 2 - 2x^2\ln(x) - 3x\ln(x)}{x(x^2 - 3x - 2)^2} \\
&= \frac{(x^2 - 3x)(1 - \ln(x)) - x^2 - 2}{x(x^2 - 3x - 2)^2}.
\end{aligned}$$

This clearly has a critical point at every root of the denominator; the only positive root is $\frac{1}{2}(3 + \sqrt{17})$. Thus this is a critical point.

It is possible to see that the numerator of the derivative is always negative, and thus the derivative is always negative. Thus $\frac{1}{2}(3 + \sqrt{17})$ is the only critical point.

(But of course this point is not in the domain of f, so it's equally valid to say that there are no critical points at all).

(c) Classify the relative extrema of $h(x) = \sqrt[3]{x}(x + 4)$

Solution: We have

$$h'(x) = \sqrt[3]{x} + \frac{1}{3}x^{-2/3}(x + 4) = \frac{x}{\sqrt[3]{x^2}} + \frac{x + 4}{3\sqrt[3]{x^2}} = \frac{4x + 4}{3\sqrt[3]{x^2}}$$

so $h'(x)$ is undefined at $x = 0$ and $h'(x) = 0$ at $x = -1$. Thus the critical points are $0, -1$. Those are the possible relative extrema.

We can classify these points in two ways. We can use the first derivative test or the second derivative test. In these solutions I'll do both.

For the second derivative test we compute:

$$\begin{aligned}
h''(x) &= \frac{4(3\sqrt[3]{x^2}) - \frac{4}{3}(x+1)\frac{-2}{3}x^{-5/3}}{9\sqrt[3]{x^4}} = \frac{12\sqrt[3]{x^2} + \frac{8}{3}(x+1)x^{-5/3}}{9\sqrt[3]{x^4}} \\
h''(-1) &= \frac{12+0}{9} = \frac{4}{3} > 0 \\
h''(0) \text{``}=\text{''} & \frac{0+0}{0} \text{ is undefined}
\end{aligned}$$

So we see that h has a local minimum at -1 since $h''(-1) > 0$, but this tells us nothing about the critical point at 0; the second derivative test is inconclusive there. So we're forced to use the first derivative test.

For the first derivative test we make a chart:

	$4x+4$	$\frac{1}{3\sqrt[3]{x^2}}$	$h'(x)$
$x < -1$	$-$	$+$	$-$
$-1 < x < 0$	$+$	$+$	$+$
$0 < x$	$+$	$+$	$+$

so h has a relative minimum at -1 and neither a maximum nor a minimum at 0.

Problem 8. (a) Find all the critical points of $g(x) = \dfrac{x^2-8}{x+3}$

Solution: The function is undefined at $x = -3$.

$g'(x) = \frac{2x(x+3) - 1(x^2-8)}{(x+3)^2} = \frac{x^2+6x+8}{(x+3)^2}$. The denominator is zero when $x = -3$, and thus the derivative is undefined there, but so is the function. The numerator is $(x+2)(x+4)$ and thus has roots when $x = -2, -4$. So the critical points of the function are -2 and -4.

(b) If $-1 \leq f'(x) \leq 3$ and $f(0) = 0$, what can you say about $f(4)$? Assume f is continuous and differentiable.

Solution: By the Mean Value Theorem, there is some c such that $f'(c) = \frac{f(4)-f(0)}{4-0}$. Since $-1 \leq f'(c) \leq 3$, we have

$$\begin{aligned}
-1 &\leq \frac{f(4)-f(0)}{4} \leq 3 \\
-4 &\leq f(4) - 0 \leq 12 \\
-4 &\leq f(4) \leq 12
\end{aligned}$$

so $f(4)$ is between -4 and 12.

(c) Prove that $x^2 - (e^2+1)\ln(x)$ has exactly two real roots.

Solution: Let $g(x) = x^2 - (e^2+1)\ln(x)$. Then g is continuous and differentiable for all real numbers greater than 0. We see that $g(1) = 1 > 0$, $g(e) = -1 < 0$, and $g(e^2) = e^4 - 2e^2 - 2 > 0$. So by the intermediate value theorem, g has a root between 1 and e, and another between e and e^2.

Now $g'(x) = 2x - \frac{e^2+1}{x}$ is zero precisely when $x^2 = \frac{e^2+1}{2}$. This equation has exactly one positive root, and g is only defined for $x > 0$, so the derivative of g is zero in exactly one place.

So suppose g has three roots, $a < b < c$. Then by Rolle's theorem (or the mean value theorem), there exists $a < x < b$ and $b < y < c$ such that $g'(x) = g'(y) = 0$. But g' has only one root, so this is impossible; thus g has exactly two roots.

Problem 9. Sketch the graph of $j(x) = x^4 - 14x^2 + 24x + 6$. (Don't worry about finding roots).

Solution: The domain of j is all reals. I'm not going to worry about finding roots now, and there are no obvious symmetries. It's a polynomial of even degree, so it's easy to see that $\lim_{x\to\pm\infty} j(x) = +\infty$.

$j'(x) = 4x^3 - 28x + 24 = 4(x^3 - 7x + 6)$ is obviously zero when $x = 1$ or $x = 2$. This very strongly suggests it has a third root; factoring (which we can do by long division or by trial-and-error) gives $j'(x) =$

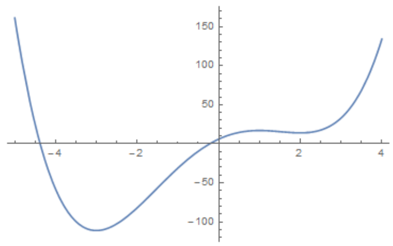
$(x+3)(x-1)(x-2)$. Thus j has three critical points, at $-3, 1, 2$. We compute j at these critical points: $j(-3) = 81 - 126 - 72 + 6 = -111$, $j(1) = 1 - 14 + 24 + 6 = 17$, $j(2) = 14$.

We can make a chart to determine when j increases or decreases:

	$(x+3)$	$(x-1)$	$(x-2)$	$j'(x)$
$x < -3$	$-$	$-$	$-$	$-$
$-3 < x < 1$	$+$	$-$	$-$	$+$
$1 < x < 2$	$+$	$+$	$-$	$-$
$2 < x$	$+$	$+$	$+$	$+$

So j is increasing between -3 and 1 and when bigger than 2, and j is decreasing when smaller than -3 or between 1 and 2. This implies that j has a relative minimum (of -111) at -3, a relative maximum (of 17) at 1, and a relative minimum of 14 at 2.

$j''(x) = 12x^2 - 28 = 4(3x^2 - 7)$ is zero when $x^2 = 7/3$, when $x = \pm\sqrt{7/3}$. $j''(x)$ is positive when $|x| > \sqrt{7/3}$ and negative when $|x| < \sqrt{7/3}$.



Graph of $j(x)$

Problem 10. Sketch a graph of the function $h(x) = e^x - \ln(|x|)$.

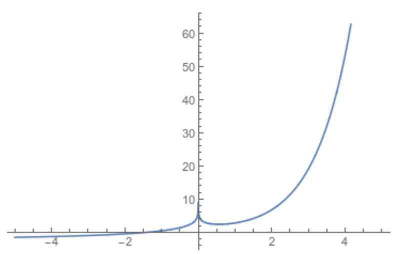
Solution: The domain of h is all real numbers except 0. $\lim_{x\to 0} e^x = 1$ and $\lim_{x\to 0} \ln(|x|) = -\infty$, so $\lim_{x\to 0} h(x) = +\infty$. The function has no obvious symmetries. $h(-1) = 1/e$ is positive and $h(-e) = 1/e^e - 1$ is negative, so h has a root between $-e$ and -1. $\lim_{x\to -\infty} h(x) = 0 - (+\infty) = -\infty$. $\lim_{x\to +\infty} h(x)$ is harder to calculuate. But

$$\lim_{x\to+\infty} h(x) = \lim_{x\to+\infty} \frac{e^{2x} - \ln(|x|)^2}{e^x + \ln(|x|)}$$

$$= \lim_{x\to+\infty} \frac{2e^{2x} - 2\frac{\ln(|x|)}{x}}{e^x + \frac{1}{x}} = \lim_{x\to+\infty} \frac{2e^{2x} - 0}{e^x + 0} = \lim_{x\to+\infty} 2e^x = +\infty.$$

$h'(x) = e^x - \frac{1}{x}$. This is undefined when $x = 0$. This has a root when $e^x = \frac{1}{x}$; since e^x is always positive this can only happen when $x > 0$. For a very small x this must be negative, but $h'(1) = e - 1 > 0$, so there is a root between 0 and 1.

It's clear that $h'(x) > 0$ when $x < 0$, since $e^x > 0$ and $-1/x > 0$. We compute $h''(x) = e^x + \frac{1}{x^2} > 0$, so the derivative is increasing everywhere; thus $h'(x)$ can have at most one root for $x > 0$. We see that h is concave up everywhere, and is increasing when $x < 0$ or when x is greater than the positive critical point; h is decreasing when x is between 0 and the positive critical point. Thus the positive critical point is a minimum.



The graph of h from -5 to 5