Math 114 Practice Test 1 Solutions

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Problem 1.

(a) Directly from the definition of a limit, compute with proof $\lim_{x \to -2} \frac{x}{x+4}$ **Solution:** We guess -1.

Let $\epsilon > 0$ and let $\delta \leq \underline{1, \epsilon/2}$. Then if $0 < |x+2| < \delta$, we compute

$$\left| \frac{x}{x+4} + 1 \right| = \left| \frac{2x+4}{x+4} \right| = \frac{2|x+2|}{|x+4|}$$

And we compute

$$|x+4| = |(x+2)+2| \geq 2 - |x+2| > 2 - \delta \geq 1$$

by the reverse triangle inequality. So

$$\left| \frac{x}{x+4} + 1 \right| = \frac{2|x+2|}{|x+4|} < \frac{2\delta}{1} < \epsilon.$$

(b) Directly from the definition, compute with proof $\lim_{x \to 3} \frac{2x^2-10x+12}{x-3}$.

Solution: Let $\epsilon > 0$ and set $\delta \leq \underline{\epsilon/2}$. Then if $0 < |x-3| < \delta$ then

$$\begin{aligned} \left| \frac{2x^2 - 10x + 12}{x-3} - 2 \right| &= \left| \frac{2(x-3)(x-2)}{x-3} - 2 \right| \\ &= |2(x-2) - 2| = 2|x-3| < 2\delta \leq \epsilon. \end{aligned}$$

Problem 2.

(a) Directly from the definition of a limit, compute with proof $\lim_{x \to 1^+} f(x)$ where

$$f(x) = \begin{cases} (x+3)^2 & x > 1 \\ x & x < 1 \end{cases}$$

Solution: We guess 16.

Let $\epsilon > 0$ and set $\delta \leq \underline{1, \epsilon/9}$. Then if $1 < x < 1 + \delta$, we have

$$\begin{aligned} |f(x) - 16| &= |(x+3)^2 - 16| = |x^2 + 6x + 9 - 16| = |x^2 + 6x - 7| \\ &= |x+7| \cdot |x-1| < |x+7|\delta \\ &= |x - 1 + 8|\delta \leq (|x-1| + 8)\delta \\ &< (\delta + 8)\delta \leq 9\delta \leq \epsilon. \end{aligned}$$

(b) Directly from the definition of a limit, prove that $\lim_{x \to -1} g(x)$ does not exist, where

$$g(x) = \begin{cases} 5 & x < -1 \\ 2 & x > -1 \end{cases}$$

Solution: Set $\epsilon = 1$ and suppose $\delta > 0$. Suppose $\lim_{x\to -1} g(x) = L$. Then set $x_1 = -1 + \delta/2, x_2 = -1 - \delta/2$, and we have

$$\begin{aligned}
\epsilon &> |g(x_1) - L| = |g(-1+\delta/2) - L| = |2 - L| \\
\epsilon &> |g(x_2) - L| = |g(-1-\delta/2) - L| = |5 - L| \\
2\epsilon &> |L-2| + |5-L| \geq |L - 2 + 5 - L| = |3| = 3
\end{aligned}$$

Thus we have $3 < 2\epsilon = 2$ which is impossible.

Problem 3.

(a) Directly from the definition, prove that $\lim_{x\to+\infty} x^2 + x + 1 = +\infty$.

Solution: Let $N > 0$ and set $M = \sqrt{N}$. Then if $x > M$, we have

$$x^2 + x + 1 > M^2 + M + 1 = N + \sqrt{N} + 1 > N.$$

(b) Directly from the definition, prove that $\lim_{x\to-2} \frac{x}{(x+2)^2} = -\infty$.

Solution: Let $N > 0$, and set $\delta \leq \underline{1, 1/\sqrt{N}}$. Then if $0 < |x+2| < \delta$, then

$$\begin{aligned}
&(x+2)^2 < \delta^2 \leq 1/N \\
&\frac{1}{(x+2)^2} > \frac{1}{\delta^2} \geq N \\
&\quad x = (x+2) - 2 < \delta - 2 \leq 1 - 2 = -1 \\
&\frac{x}{(x+2)^2} < -N && \text{(sign flips because } -1 < 0).
\end{aligned}$$

Problem 4.

Compute the following limits, showing each step and naming each limit law you use.

(a)

$$\lim_{x\to 4} \sqrt{x^2 - x - 3} + \frac{2}{x}$$

Solution:

$$\begin{aligned}
\lim_{x\to 4} \sqrt{x^2 - x - 3} + \frac{2}{x} &= \lim_{x\to 4} \sqrt{x^2 - x - 3} + \lim_{x\to 4} \frac{2}{x} && \text{Additivity} \\
&= \sqrt{\lim_{x\to 4} x^2 - x - 3} + \lim_{x\to 4} \frac{2}{x} && \text{Exponents} \\
&= \sqrt{\lim_{x\to 4} x^2 - \lim_{x\to 4} x - \lim_{x\to 4} 3} + \lim_{x\to 4} \frac{2}{x} && \text{Additivity} \\
&= \sqrt{\left(\lim_{x\to 4} x\right)^2 - \lim_{x\to 4} x - \lim_{x\to 4} 3} + \lim_{x\to 4} \frac{2}{x} && \text{Exponents} \\
&= \sqrt{\left(\lim_{x\to 4} x\right)^2 - \lim_{x\to 4} x - \lim_{x\to 4} 3} + \frac{\lim_{x\to 4} 2}{\lim_{x\to 4} x} && \text{Quotients} \\
&= \sqrt{(4)^2 - 4 - \lim_{x\to 4} 3} + \frac{\lim_{x\to 4} 2}{4} && \text{Identity} \\
&= \sqrt{(4)^2 - 4 - 3} + \frac{2}{4} && \text{Constants} \\
&= \sqrt{16 - 4 - 3} + \frac{2}{4} = 3 + \frac{1}{2} && \text{Arithmetic}
\end{aligned}$$

(b)

$$\lim_{x\to 1} \frac{x^2+4x-5}{x-1}$$

Solution:

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2+4x-5}{x-1} &= \lim_{x\to 1} \frac{(x+5)(x-1)}{x-1} && \text{Arithmetic} \\
&= \lim_{x\to 1} x+5 && \text{Almost Identical Functions} \\
&= \lim_{x\to 1} x + \lim_{x\to 1} 5 && \text{Additivity} \\
&= 1 + \lim_{x\to 1} 5 && \text{Identity} \\
&= 1+5 && \text{Constants} \\
&= 6
\end{aligned}$$

Problem 5.

Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x\to -2} \frac{x^2+6x+8}{2(x+4)(x+2)} =$$

Solution:

$$\lim_{x\to -2} \frac{x^2+6x+8}{2(x+4)(x+2)} = \lim_{x\to -2} \frac{(x+4)(x+2)}{2(x+4)(x+2)} = \lim_{x\to -2} \frac{1}{2} = 1/2.$$

(b)

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x}$$

Solution:

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x} = \lim_{x\to 9} \frac{(3-\sqrt{x})(3+\sqrt{x})}{(9-x)(3+\sqrt{x})} = \lim_{x\to 9} \frac{9-x}{(9-x)(3+\sqrt{x})} = \lim_{x\to 9} \frac{1}{3+\sqrt{x}} = 1/6.$$

(c)

$$\lim_{x\to -\infty} \frac{3x^3+\sqrt[3]{x}}{\sqrt{9x^6+2x^2+1}+x}$$

Solution:

$$\begin{aligned}
\lim_{x\to -\infty} \frac{3x^3+\sqrt[3]{x}}{\sqrt{9x^6+2x^2+1}+x} &= \lim_{x\to -\infty} \frac{3x^3/x^3+\sqrt[3]{x}/x^3}{\sqrt{9x^6+2x^2+1}/(-\sqrt{x^6})+x/x^3} \\
&= \lim_{x\to -\infty} \frac{3+x^{-8/3}}{-\sqrt{9+2x^{-4}+x^{-6}}+x^{-2}} \\
&= \lim_{x\to -\infty} \frac{3}{-\sqrt{9}} = -1.
\end{aligned}$$

(d)

$$\lim_{x\to 1^+} \frac{|x-1|}{x-1} =$$

Solution: We note that when $x>1$, $|x-1| = x-1$, so we have

$$\lim_{x\to 1^+} \frac{|x-1|}{x-1} = \lim_{x\to 1^+} \frac{x-1}{x-1} = \lim_{x\to 1^+} 1 = 1$$