Math 114 Practice Test 2 Solutions

Instructor: Jay Daigle

Problem 1. Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} =$$

Solution:

$$\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} = \lim_{x\to 1} \left(\frac{\sin(x-1)}{x-1}\right)^2 = \left(\lim_{x\to 1} \frac{\sin(x-1)}{x-1}\right)^2 = 1^2 = 1$$

by the small angle approximation.

(b)

$$\lim_{x\to -2} \frac{x^2+6x+9}{2(x+4)(x+2)} =$$

Solution: We know that $\lim_{x\to -2} x^2+6x+9 = 1$ and $\lim_{x\to -2} 2(x+4)(x+2) = 0$. So

$$\lim_{x\to -2} \frac{x^2+6x+9}{2(x+4)(x+2)} = \pm\infty.$$

Since $2(x+4)(x+2)$ can be either positive or negative near -2—it is negative for values just less than -2 and positive for values just greater—we can't do any better than this.

(c) Using the Squeeze Theorem, show that

$$\lim_{x\to 3} \frac{x-3}{1+\sin^2\left(\frac{2\pi+e+7}{x-3}\right)} = 0.$$

Solution: Observe that since $-1 \leq \sin(a) \leq 1$ for any a, we have that $0 \leq \sin^2(a) \leq 1$ for any a, and thus $1 \leq 1+\sin^2(a) \leq 2$. Taking the reciprocal gives us $1/2 \leq \frac{1}{1+\sin^2(a)} \leq 1$ for any a, and in particular for $a = \frac{2\pi+e+7}{x-3}$. Taking absolute values and multiplying by $|x-3|$ gives

$$\left|\frac{x-3}{2}\right| \leq \left|\frac{x-3}{1+\sin^2\left(\frac{2\pi+e+7}{x-3}\right)}\right| \leq |x-3|.$$

By continuity, we can compute that $\lim_{x\to 3} |(x-3)/2 = \lim_{x\to 3} |x-3| = 0$. So by the squeeze theorem we know that

$$\lim_{x\to 3} \left|\frac{x-3}{1+\sin^2\left(\frac{2\pi+e+7}{x-3}\right)}\right| = 0,$$

and thus

$$\lim_{x\to 3} \frac{x-3}{1+\sin^2\left(\frac{2\pi+e+7}{x-3}\right)} = 0.$$

Problem 2. (a) Show that the polynomial $x^4 - 6x - 2$ has two real roots, that is, there are two (different!) real numbers a and b such that $a^4 - 6a - 2 = b^4 - 6b - 2 = 0$.

Solution: Set $f(x) = x^4 - 6x - 2$; since this is a polynomial function it must be continuous. We compute:

$$f(0) = -2 \qquad f(-1) = 5$$
$$f(1) = -7 \qquad f(2) = 2$$

We have $-2 < 0 < 5$, so by the Intermediate Value Theorem there is some a between -1 and 0 with $f(a) = 0$. Similarly, we have $-7 < 0 < 2$ so by the Intermediate Value theorem there is some b between 1 and 2 with $f(b) = 0$. Clearly a and b are different since $a < 0$ and $b > 1$, so a and b are two distinct roots to the polynomial $x^4 - 6x - 2$.

(b) Let

$$g(x) = \begin{cases} \frac{x^2-1}{x-1} & x > 0 \\ x^2 + 1 & x < 0 \end{cases}$$

If possible, define an extension of g that is continuous at all real numbers. **Solution:** g fails to be defined at 2 placs: 0 and 1. We see that

$$\lim_{x\to 1} g(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x + 1 = 2$$

so we wish to set $g_F(1) = 2$. (Alternatively, we can just replace the $\frac{x^2-1}{x-1}$ with an $x + 1$).

At 0, we need to compute the two one-sided limits. We have

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} x^2 + 1 = 1$$
$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{x^2-1}{x-1} = \frac{-1}{-1} = 1.$$

Thus the discontinuity is removeable, and we want to set $g_F(0) = 1$. Thus our continuous extension is

$$g_F(x) = \begin{cases} x+1 & x > 0 \\ 1 & x = 0 \\ x^2+1 & x < 0 \end{cases} = \begin{cases} x+1 & x \geq 0 \\ x^2+1 & x \leq 0 \end{cases}$$

Problem 3. Compute the following derivatives using only the definition of derivative.

(a) Derivative of $f(x) = x^2 + \sqrt{x}$ at $x = 2$.

Solution:

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0} \frac{(2+h)^2 + \sqrt{2+h} - 2^2 - \sqrt{2}}{h} \\ &= \left(\lim_{h\to 0} \frac{4h + h^2}{h}\right) + \left(\lim_{h\to 0} \frac{(\sqrt{2+h} - \sqrt{2})(\sqrt{2+h} + \sqrt{2})}{h(\sqrt{2+h} + \sqrt{2})}\right) \\ &= \left(\lim_{h\to 0} 4 + h\right) + \left(\lim_{h\to 0} \frac{1}{\sqrt{2+h} + \sqrt{2}}\right) \\ &= 4 + \frac{1}{2\sqrt{2}}. \end{aligned}$$

(b) Derivative of $g(x) = \frac{1}{x+1}$ at $x = 1$.

Solution:

$$\begin{aligned}
g'(1) &= \lim_{h\to 0} \frac{g(1+h) - g(1)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2}{4+2h} - \frac{2+h}{4+2h}}{h} \\
&= \lim_{h\to 0} \frac{-h}{h(4+2h)} \\
&= \lim_{h\to 0} \frac{-1}{4+2h} \\
&= \frac{-1}{4}.
\end{aligned}$$

Problem 4. You may use any methods we have learned in class to solve these problems, but show enough work to justify your answers.

(a) Find $\frac{d^2 f}{dx^2}$ if $f(x) = x\cos x$.

Solution:

$$\begin{aligned}
f'(x) &= \cos x - x\sin x \\
f''(x) &= -\sin(x) - \sin(x) - x\cos x \\
&= -2\sin(x) - x\cos x
\end{aligned}$$

(b) If $g(x) = \sin(3x)$ compute $g'(\pi/12)$

Solution:

$$\begin{aligned}
g'(x) &= \cos(3x)\cdot 3. \\
g'(\pi/12) &= \cos(\pi/4)\cdot 3 = 3\sqrt{2}/2
\end{aligned}$$

(c) Find an equation of the line tangent to $y = \frac{x^2-1}{x^2+1}$ at the point $(0, -1)$.

Solution: We have that

$$y' = \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2}$$

so when $x = 0$ we have $y' = (0-0)/1 = 0$. The equation for a tangent line is $y = m(x - x_0) + y_0$, so the tangent line to this function at $(0, 1)$ is $y = 0(x-0) + (-1)$, or $y = -1$.