Math 114 Practice Exam 3 Solutions

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Problem 1. Compute the derivatives of the following functions using methods we have learned in class. Show enough work to justify your answers.

(a) $f(x) = \sec\left(\frac{\sqrt{x^2+1}}{e^x+2}\right)$

Solution:

$$f'(x) = \sec\left(\frac{\sqrt{x^2+1}}{e^x+2}\right)\cdot\tan\left(\frac{\sqrt{x^2+1}}{e^x+2}\right)\cdot\frac{\frac{1}{2}(x^2+1)^{-1/2}2x(e^x+2)-e^x\sqrt{x^2+1}}{(e^x+2)^2}$$

(b) $g(x) = \sqrt[4]{\frac{x^3+\cos(x^2)}{\log_3(x+1)+1}}$

Solution:

$$g'(x) = \frac{1}{4}\left(\frac{x^3+\cos(x^2)}{\log_3(x+1)+1}\right)^{-3/4}\cdot\frac{(3x^2-\sin(x^2)2x)(\log_3(x+1)+1)-\frac{1}{(x+1)\ln(3)}(x^3+\cos(x^2))}{(\log_3(x+1)+1)^2}$$

Problem 2. (a) Find a formula for y' in terms of x and y if $x^8+x^4+y^4+y^6=1$.

Solution:

$$\begin{aligned} 8x^7+4x^3+4y^3\frac{dy}{dx}+6y^5\frac{dy}{dx} &= 0 \\ 8x^7+4x^3 &= (4y^3+6y^5)\frac{dy}{dx} \\ -\frac{4x^7+2x^3}{2y^3+3y^5} &= \frac{dy}{dx}. \end{aligned}$$

(b) Compute $f'(\pi)$ where $f(x)=3^{\sin(x)}$.

Solution:

$$\begin{aligned} f'(x) &= 3^{\sin(x)}\ln(3)\cos(x) \\ f'(\pi) &= 3^{\sin(\pi)}\ln(3)\cos(\pi) = 3^0\ln(3)(-1) = -\ln(3). \end{aligned}$$

(c) Compute $g'(4)$ where $g(x)=\ln(x^3+3x+\sqrt{x})$.

Solution:

$$g'(x) = \frac{1}{x^3+3x+\sqrt{x}}\left(3x^2+3+\frac{1}{2\sqrt{x}}\right)$$

so

$$g'(4) = \frac{1}{4^3+3\cdot 4+\sqrt{4}}\left(3(4^2)+3+\frac{1}{2\sqrt{4}}\right) = \frac{51+\frac{1}{4}}{78} = \frac{205}{312}.$$

Problem 3. (a) Let $j(x) = \sqrt[3]{x^5 + x^4 + x^3 + x^2 + 2x}$. Find $(j^{-1})'(4)$.

Solution: Plugging in numbers, we see that $j(2) = \sqrt[3]{32 + 16 + 8 + 4 + 4} = \sqrt[3]{64} = 4$. Then by the Inverse Function Theorem we have $(j^{-1})'(4) = \frac{1}{j'(2)}$. But

$$j'(x) = \frac{1}{3}\left(x^5 + x^4 + x^3 + x^2 + 2x\right)^{-2/3}\left(5x^4 + 4x^3 + 3x^2 + 2x + 2\right)$$
$$j'(2) = \frac{1}{3}(64)^{-2/3}(80 + 32 + 12 + 4 + 2) = \frac{130}{48} = \frac{65}{24}.$$

Thus by the inverse function theorem we have

$$(j^{-1})'(4) = \frac{24}{65}.$$

(b) Find a tangent line to the curve given by $x^4 - 2x^2y^2 + y^4 = 16$ at the point $(\sqrt{5}, 1)$.

Solution: We use implicit differentiation, and find that

$$4x^3 - 2\left((2xy^2 + x^2 2y\frac{dy}{dx}\right) + 4y^3\frac{dy}{dx} = 0$$
$$4x^3 - 4xy^2 = 4x^2y\frac{dy}{dx} - 4y^3\frac{dy}{dx}$$
$$\frac{4x^3 - 4xy^2}{4x^2y - 4y^3} = \frac{dy}{dx}$$

Thus at the point $(\sqrt{5}, 1)$ we have

$$\frac{dy}{dx} = \frac{4\sqrt{5}^3 - 4\sqrt{5}\cdot 1^2}{4\sqrt{5}^2\cdot 1 - 4\cdot 1^3} = \sqrt{5}\left(\frac{20-4}{20-4}\right) = \sqrt{5}.$$

Thus the equation of our tangent line is

$$y - y_0 = m(x - x_0)$$
$$y - 1 = \sqrt{5}(x - \sqrt{5}).$$

Problem 4. (a) It is a fact that $2^{10} = 1024$. Estimate 2.01^{10} using the derivative of x^{10} at the point 2.

Solution: We set $f(x) = x^{10}$ and see that $f'(x) = 10x^9$. Using our formulas, we then have

$$f(2.01) \approx f(2) + (2.01 - 2)f'(2)$$
$$= 1024 + .01(10\cdot 2^9) = 1024 + 512/10 = 1024 + 51.2 = 1075.2.$$

(b) Suppose we have the differential equation $f'(t) = f(t) - t$, with $f(1) = 2$. Use Euler's method with three steps to approximate $f(4)$.

Solution: We have

$$f(2) \approx f'(1)(2-1) + f(1) = 1 + 2 = 3$$
$$f(3) \approx f'(2)(3-2) + f(2) \approx 1(3-2) + 3 = 4$$
$$f(4) \approx f'(3)(4-3) + f(3) \approx 1(4-3) + 4 = 5.$$

(c) Use two iterations of Newton's method, starting at 2, to estimate the cube root of 9.

Solution: Let $f(x) = x^3 - 9$ so that $f(\sqrt[3]{9}) = 0$. Then we compute $f'(x) = 3x^2$ so

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{-1}{12} = \frac{25}{12}$$
$$x_3 = \frac{25}{12} - \frac{\left(\frac{25}{12}\right)^3 - 9}{3\left(\frac{25}{12}\right)^2}.$$

(We can work out that this is equal to $\frac{23401}{11250}$ or approximately 2.08).

Problem 5. (a) Solve a stereotypical math problem: Two cars start moving from the same point. One travels south at 55 miles per hour and the other travels west at 30 miles per hour. Two hours later, how quickly is the distance between the two cars increasing?

Solution: Write a for the distance the first car is south of the origin, and b for the distance the second car is west of the origin. Then we have $a = 110, a' = 55, b = 60, b' = 30$. We write the equation $d^2 = a^2 + b^2$ to relate the distance to things we already know. We compute that $d^2 = 110^2 + 60^2$ and thus $d = \sqrt{110^2 + 60^2} = 10\sqrt{11^2 + 6^2}$.

Taking the derivative gives us

$$
\begin{aligned}
2dd' &= 2aa' + 2bb' \\
d' &= d^{-1}(110 \cdot 55 + 60 \cdot 30) \\
&= 50\frac{11^2 + 6^2}{10\sqrt{11^2 + 6^2}} = 5\sqrt{157}.
\end{aligned}
$$

(b) A car is driving down a road at 150 feet per second (this is about a hundred miles an hour). A camera is placed 200 feet from the road, which will rotate to follow and record the progress of the car. How quickly must the camera rotate when the car is fifty feet away from directly in front of the camera?

Solution: Let the car's position be x, and the angle at which the camera is pointing is θ. Then we have $x = 50, x' = 150$, and we are looking for θ'. We have the equation $\tan\theta = x/200$, and thus

$$
\begin{aligned}
\sec^2\theta \cdot \theta' &= \frac{x'}{200} \\
&= 150/200 = 3/4.
\end{aligned}
$$

But we know that our triangle has sides of length 50 and 200, so the hypotenuse must have length $\sqrt{50^2 + 200^2} = \sqrt{2500 + 40000} = \sqrt{42500} = 10\sqrt{425} = 50\sqrt{17}$. Thus $\sec\theta = \sqrt{17}/4$ and $\sec^2\theta = (\sqrt{17}/4)^2 = 17/16$, and we have

$$
\begin{aligned}
17\theta'/16 &= 3/4 \\
\theta' &= 12/17 \approx .70588.
\end{aligned}
$$

Thus the camera must rotate at 12/17 radians/second.