Math 114 Test 1 Solutions

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Problem 1.

(a) Directly from the definition of a limit, compute with proof $\lim_{x \to 3} \frac{x+1}{2}$

Solution: We guess 2.

Let $\epsilon > 0$ and let $\delta \leq \underline{2\epsilon}$. Then if $0 < |x-3| < \delta$, we compute

$$\left| \frac{x+1}{2} - 2 \right| = \left| \frac{x+1-4}{2} \right| = \frac{1}{2}|x-3| < \delta/2 = \epsilon.$$

(b) Directly from the definition, compute with proof $\lim_{x \to 1} \frac{x-3}{x+2}$.

Solution: We guess $-2/3$.

Let $\epsilon > 0$ and set $\delta \leq \underline{1, 6\epsilon/5}$. Then if $0 < |x-1| < \delta$ then

$$\begin{aligned} \left| \frac{x-3}{x+2} + 2/3 \right| &= \left| \frac{3(x-3)+2(x+2)}{3(x+2)} \right| = \left| \frac{5x-5}{3(x+2)} \right| \\ &= \frac{5|x-1|}{3|x+2|} < \frac{5\delta}{3|x+2|}. \end{aligned}$$

Now we compute that

$$|x+2| = |x-1+3| \geq 3 - |x-1| > 3 - \delta \geq 2$$

so

$$\frac{5\delta}{3|x+2|} < \frac{5\delta}{3 \cdot 2} = \frac{5}{6}\delta \leq \epsilon.$$

Problem 2.

(a) Directly from the definition of a limit, compute with proof $\lim_{x \to -1^-} f(x)$ where

$$f(x) = \begin{cases} 5x^2 - 17\sqrt{x} & x > -1 \\ 3x+1 & x < -1 \end{cases}$$

Solution: We guess -2.

Let $\epsilon > 0$ and set $\delta \leq \underline{\epsilon/3}$. Then if $-1-\delta < x < -1$, we have

$$|f(x)+2| = |3x+1+2| = 3|x+1| < 3\delta \leq \epsilon.$$

(b) Directly from the definition of a limit, prove that $\lim_{x \to 5} g(x)$ does not exist, where

$$g(x) = \begin{cases} 1 & x < 5 \\ 3 & x > 5 \end{cases}$$

Solution: Set $\epsilon = 1$ and suppose $\delta > 0$. Suppose $\lim_{x\to 5} g(x) = L$. Then set $x_1 = 5+\delta/2, x_2 = 5-\delta/2$, and we have

$$\begin{aligned}\epsilon &> |g(x_1) - L| = |g(5+\delta/2) - L| = |3 - L|\\ \epsilon &> |g(x_2) - L| = |g(5-\delta/2) - L| = |1 - L|\\ 2\epsilon &> |L-3| + |1-L| \geq |L-3+1-L| = |-2| = 2\end{aligned}$$

Thus we have $2 < 2\epsilon = 2$ which is impossible.

Problem 3.

(a) Directly from the definition, prove that $\lim_{x\to -\infty} \frac{2x}{x+1} = 2$.

Solution: Let $\epsilon > 0$ and set $M = \underline{2/\epsilon + 1}$. Then if $x < -M$, we have

$$\left|\frac{2x}{x+1} - 2\right| = \left|\frac{2x - 2(x+1)}{x+1}\right| = \left|\frac{-2}{x+1}\right| = \frac{2}{|x+1|}.$$

We compute $x + 1 < 1 - M < 0$, so $|x+1| > |M-1| = M - 1 = 2/\epsilon$. Then we have

$$\frac{2}{|x+1|} < \frac{2}{2/\epsilon} = \epsilon.$$

(b) Directly from the definition, prove that $\lim_{x\to 3} \frac{x}{x-3} = \pm\infty$.

Solution: Let $N > 0$, and set $\delta \leq \underline{1, 2/N}$. Then if $0 < |x-3| < \delta$, then

$$\begin{aligned}\left|\frac{x}{x-3}\right| &= \frac{|x|}{|x-3|} > \frac{|x|}{\delta} = \frac{|x-3+3|}{\delta}\\ &\geq \frac{3 - |x-3|}{\delta} > \frac{2}{\delta} > \frac{2}{2/N} = N.\end{aligned}$$

Problem 4.

Compute the following limits, showing each step and naming each limit law you use. Use only one law per line!

(a)

$$\lim_{x\to 0} \frac{x^2 + \sqrt{x+4} - 1}{x+3}$$

Solution:

$$\begin{aligned}\lim_{x\to 0} \frac{x^2 + \sqrt{x+4} - 1}{x+3} &= \frac{\lim_{x\to 0} x^2 + \sqrt{x+4} - 1}{\lim_{x\to 0} x + 3} && \text{Quotients}\\ &= \frac{\lim_{x\to 0} x^2 + \lim_{x\to 0}\sqrt{x+4} - \lim_{x\to 0} 1}{\lim_{x\to 0} x + \lim_{x\to 0} 3} && \text{Additivity}\\ &= \frac{(\lim_{x\to 0} x)^2 + \sqrt{\lim_{x\to 0} x + 4} - \lim_{x\to 0} 1}{\lim_{x\to 0} x + \lim_{x\to 0} 3} && \text{exponents}\\ &= \frac{(\lim_{x\to 0} x)^2 + \sqrt{\lim_{x\to 0} x + \lim_{x\to 0} 4} - \lim_{x\to 0} 1}{\lim_{x\to 0} x + \lim_{x\to 0} 3} && \text{additivity}\\ &= \frac{0^2 + \sqrt{0+4} - 1}{0+3} = \frac{1}{3} && \text{Identity and Constants}\end{aligned}$$

(b)

$$\lim_{x\to 2} \frac{x^2 + -7x + 10}{x - 2}$$

Solution:

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2 + -7x + 10}{x - 2} &= \lim_{x\to 2} \frac{(x-2)(x-5)}{x-2} \\
&= \lim_{x\to 2} (x - 5) && \text{Almost Identical Functions} \\
&= \lim_{x\to 2} x - \lim_{x\to 2} 5 && \text{Additivity} \\
&= 2 - 5 = -3 && \text{Identity and Constants}
\end{aligned}$$

Problem 5.

Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x\to 4} \frac{x^3 - 64}{x - 4} =$$

Solution:

$$\lim_{x\to 4} \frac{x^3 - 64}{x - 4} = \lim_{x\to 4} \frac{(x-4)(x^2 + 4x + 16)}{x - 4} = \lim_{x\to 4} x^2 + 4x + 16 = 48.$$

(b)

$$\lim_{x\to 7} \frac{\sqrt{9 + x} - 4}{x - 7}$$

Solution:

$$\lim_{x\to 7} \frac{\sqrt{9 + x} - 4}{x - 7} = \lim_{x\to 7} \frac{9 + x - 16}{(x - 7)(\sqrt{9 + x} + 4)} = \lim_{x\to 7} \frac{1}{\sqrt{9 + x} + 4} = \frac{1}{8}.$$

(c)

$$\lim_{x\to +\infty} \frac{x^2 + x + 1}{\sqrt{x^4 + 1}}$$

Solution:

$$\lim_{x\to +\infty} \frac{x^2 + x + 1}{\sqrt{x^4 + 1}} = \lim_{x\to +\infty} \frac{1 + 1/x + 1/x^2}{\sqrt{1 + 1/x^4}} = 1.$$

(d) If

$$f(x) = \begin{cases} x^2 - 1 & x < 3 \\ \sqrt{6 + x} + 6 & x > 3 \end{cases}$$

then

$$\lim_{x\to 3^+} f(x) =$$

Solution: We note that when $x > 3$, then $f(x) = \sqrt{6 + x} + 6$, so

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \sqrt{6 + x} + 6 = \sqrt{6 + 3} + 6 = 9.$$