Math 114 Test 3 Solutions

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Problem 1. Compute the derivatives of the following functions using methods we have learned in class. Show enough work to justify your answers.

(a) $f(x) = \exp\left(\sec\left(\frac{\sqrt{x^2+1}+\log_5(x)}{x^4-\sin(x)}\right)\right)$

Solution:

$$f'(x) = \exp\left(\sec\left(\frac{\sqrt{x^2+1}+\log_5(x)}{x^4-\sin(x)}\right)\right)\sec\left(\frac{\sqrt{x^2+1}+\log_5(x)}{x^4-\sin(x)}\right)\tan\left(\frac{\sqrt{x^2+1}+\log_5(x)}{x^4-\sin(x)}\right)$$
$$\cdot\frac{\left(\frac{1}{2}(x^2+1)^{-1/2}2x+\frac{1}{x\ln(5)}\right)(x^4-\sin(x))-\left(4x^3-\cos(x)\right)\left(\sqrt{x^2+1}+\log_5(x)\right)}{(x^4-\sin(x))^2}.$$

(b) $g(x) = \sin^4\left(\tan\left(\ln\left(x^2+1\right)\right)\right)$

Solution:

$$g'(x) = 4\sin^3\left(\tan\left(\ln\left(x^2+1\right)\right)\right)\cos\left(\tan\left(\ln\left(x^2+1\right)\right)\right)\sec^2\left(\ln\left(x^2+1\right)\right)\frac{1}{x^2+1}2x.$$

Problem 2. (a) Find a formula for y' in terms of x and y if $xy^3 = \sqrt{x^2+y^2}$.

Solution: Using implicit differentiation, we have

$$\begin{aligned}
y^3 + 3xy^2y' &= \frac{2x+2yy'}{2\sqrt{x^2+y^2}} \\
&= \frac{x+yy'}{\sqrt{x^2+y^2}} \\
y^3 - \frac{x}{\sqrt{x^2+y^2}} &= \frac{yy'}{\sqrt{x^2+y^2}} - 3xy^2y' \\
y' &= \frac{y^3-\frac{x}{\sqrt{x^2+y^2}}}{\frac{y}{\sqrt{x^2+y^2}}-3xy^2} \\
&= \frac{y^3\sqrt{x^2+y^2}-x}{y-3xy^2\sqrt{x^2+y^2}}.
\end{aligned}$$

(b) Compute $f'(5)$ where $f(x) = \log_3(\sqrt{x^2+2})$.

Solution:

$$\begin{aligned}
f'(x) &= \frac{\frac{1}{2}(x^2+2)^{-1/2}2x}{\sqrt{x^2+2}\ln(3)} = \frac{x}{(x^2+2)\ln(3)} \\
f'(5) &= \frac{5}{27\ln(3)}.
\end{aligned}$$

Alternatively, we can note that $f(x) = \frac{1}{2}\log_3(x^2+2)$ and then

$$f'(x) = \frac{1}{2}\frac{2x}{(x^2+2)\ln(3)} = \frac{x}{(x^2+2)\ln(3)}$$
$$f'(5) = \frac{5}{27\ln(3)}.$$

(c) Compute $g'(4)$ where $g(x) = 2^{\sqrt{x}}$.

Solution:

$$g'(x) = 2^{\sqrt{x}}\ln(2)\frac{1}{2}x^{-1/2}$$

so

$$g'(4) = 2^2\ln(2)\frac{1}{2\cdot 2} = \ln(2).$$

Problem 3. (a) Let $j(x) = \sqrt{x^5+3x^3++5x}$. Find $(j^{-1})'(3)$.

Solution: Plugging in numbers, we see that $j(1) = \sqrt{1+3+5} = \sqrt{9} = 3$, so $j^{-1}(3) = 1$. Then by the Inverse Function Theorem we have $(j^{-1})'(3) = \frac{1}{j'(1)}$. But

$$j'(x) = \frac{1}{2}(x^5+3x^3+5x)^{-1/2}(5x^4+9x^2+5)$$
$$j'(1) = \frac{1}{2}(9)^{-1/2}(5+9+5) = \frac{19}{6}.$$

Thus by the inverse function theorem we have

$$(j^{-1})'(3) = \frac{6}{19}.$$

(b) Find a tangent line to the curve given by $x^2y + (y-x)^2 = 5$ at the point $(2,1)$.

Solution: We use implicit differentiation, and find that

$$2xy + x^2y' + 2(y-x)(y'-1) = 0$$
$$2xy + x^2y' + 2yy' - 2xy' - 2y + 2x = 0$$
$$2xy - 2y + 2x = 2xy' - 2yy' - x^2y'$$
$$y' = \frac{2xy-2y+2x}{2x-2y-x^2}$$

Thus at the point $(2,1)$ we have

$$y' = \frac{4-2+4}{4-2-4} = \frac{6}{-2} = -3$$

Thus the equation of our tangent line is

$$y - y_0 = m(x-x_0)$$
$$y - 1 = -3(x-2).$$

Alternatively, we could have computed

$$2xy + x^2y' + 2(y-x)(y'-1) = 0$$

and then substituted in, getting

$$\begin{aligned}4 + 4y' + 2(1-2)(y'-1) &= 0\\ 4 + 4y' - 2y' + 2 &= 0\\ 2y' &= -6\\ y' &= -3\end{aligned}$$

giving the same result.

Problem 4. (a) Use a tangent line approximation to estimate $\sqrt{7}$ using the derivative of $\sqrt{x}$ at the point 9.

Solution: We set $f(x) = \sqrt{x}$ and see that $f'(x) = \frac{1}{2\sqrt{x}}$. Using our formulas, we then have

$$\begin{aligned}f(7) &\approx f(9) + (7-9)f'(9)\\ &= 3 + (-2)\frac{1}{6} = 3 - \frac{1}{3} = \frac{8}{3}.\end{aligned}$$

(b) Suppose we have the differential equation $f'(t) = \frac{f(t)^2}{2} - t$, with $f(0) = 1$. Use Euler's method with three steps to approximate $f(3)$.

Solution: We have

$$\begin{aligned}f(1) &\approx f'(0)(1-0) + f(0) = \frac{1}{2} + 1 = \frac{3}{2}\\ f(2) &\approx f'(1)(2-1) + f(1) \approx \frac{1}{8} + \frac{3}{2} = \frac{13}{8}\\ f(3) &\approx f'(2)(3-2) + f(2) \approx \left(\frac{169}{128} - 2\right) + \frac{13}{8} = \frac{121}{128} \approx .95\end{aligned}$$

(c) Use two iterations of Newton's method, starting at -1 to find a root of $g(x) = x^3 + x^2 + 1$

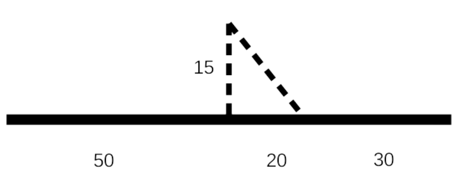
Solution: We compute $g'(x) = 3x^2 + 2x$, so

$$\begin{aligned}x_2 &= x_1 - \frac{g(x_1)}{g'(x_1)} = -1 - \frac{1}{1} = -2\\ x_3 &= -2 - \frac{-3}{8} = \frac{-13}{8}.\end{aligned}$$

Problem 5. (a) Spilled water is spreading in a circle whose area is increasing by 10π cm^2 per second. How fast is the radius increasing when the radius is 5 cm? (Recall the area of a circle is given by $A = \pi r^2$).

Solution: We have $A = \pi r^2$, so $A' = 2\pi r r'$. When the radius is 5 cm we have $10\pi = A' = 10\pi r'$ and thus $r' = 1$.

(b) A spectator is watching Usain Bolt run a 100m race. The spectator sits at the midpoint of the track (50m from each end), and 15 m away from the track. Usain Bolt runs 12m/s when he's at the 70m mark (yes, really). If the spectator is watching Bolt, how quickly is he rotating (how quickly is his angle from the track changing) when Bolt is at the 70m mark?



Solution: Write B for Usain Bolt's distance from the midpoint of the track, and θ for the angle between the spectator and the track. Then we have that $B = 20$ and $B' = 12$. We don't know θ but we can compute $\tan\theta$ easily since it's the opposite side over the adjacent side; thus $\tan\theta = B/15$. Then

$$\begin{aligned} \tan\theta &= B/15 \\ \sec^2\theta \cdot \theta' &= B'/15 \\ \theta' &= \cos^2(\theta) \cdot 12/15. \end{aligned}$$

We can compute that $\cos(\theta) = 15/25 = 3/5$, so we have

$$\theta' = \frac{9}{25} \cdot \frac{4}{5} = \frac{36}{125}.$$

Thus the spectator is rotating at $\frac{36}{125}$ radians per second; eyeballing this, he will turn 180° in about 10 seconds, which seems about right since that's how long it takes Bolt to run the 100 m.