Math 114 Test 2 Solutions

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Problem 1. Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x\to 3} \frac{(x-5)(x-1)}{(x-3)^2} =$$

Solution: $\lim_{x\to 3}(x-5)(x-1) = -4 \neq 0$ and $\lim_{x\to 3}(x-3)^2 = 0$, so the limit is $\pm infty$. The top is always negative and the bottom is always positive, so in fact the limit is $-\infty$.

(b)

$$\lim_{x\to 0} \frac{x\sin(x)}{\sin(2x)\sin(3x)} =$$

Solution:

$$\lim_{x\to 0} \frac{x\sin(x)}{\sin(2x)\sin(3x)} = \lim_{x\to 0} \frac{2x}{\sin(2x)}\frac{\sin(x)}{x}\frac{3x}{\sin(3x)}\frac{1}{6} = 1/6.$$

(c) Using the Squeeze Theorem, show that

$$\lim_{x\to -1}(x+1)\sin\left(\frac{1}{x+1}\right) = 0.$$

Solution: We have

$$-1\&leq \sin\left(\frac{1}{1+x}\right) \leq 1$$

$$0 \leq \left|\sin\left(\frac{1}{x+1}\right)\right| \leq 1$$

$$0 \leq \left|(x+1)\sin\left(\frac{1}{x+1}\right)\right| \leq |x+1|.$$

Since $\lim_{x\to -1} 0 = \lim_{x\to -1}|x+1| = 0$, by the squeeze theorem we know that

$$\lim_{x\to -1}\left|(x+1)\sin\left(\frac{1}{x+1}\right)\right| = 0$$

$$\lim_{x\to -1}(x+1)\sin\left(\frac{1}{x+1}\right) = 0.$$

Problem 2. (a) Let $f(x) = x^2 + \sin(x)$. Prove that there is a real number c such that $f(c) = 5$.

Solution:

f is continuous at all reals, so we just need to find outputs that are larger and smaller than 5. We see that $f(0) = 0$, and $f(3) = 9 + \sin(3) \geq 8 > 5$. Thus by the intermediate value theorem, there is some c in $(0,3)$ such that $f(c) = 5$.

(b) Let

$$g(x) = \begin{cases} x^3 & x < 2 \\ 10 - x & x > 2 \end{cases}.$$

Define an extension of g that is continuous at all real numbers.

Solution: We compute

$$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} x^3 = 8$$
$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} 10 - x = 8.$$

So we define

$$g_F(x) = \begin{cases} x^3 & x < 2 \\ 10 - x & x > 2 \\ 8 & x = 2 \end{cases} = \begin{cases} x^3 & x \leq 2 \\ 10 - x & x \geq 2 \end{cases}.$$

Problem 3. Compute the following derivatives using only the definition of derivative.

(a) Derivative of $f(x) = x^3$ at $a = 1$.

Solution:

$$f'(1) = \lim_{h \to 0} \frac{(1+h)^3 - 1^3}{h} = \lim_{h \to 0} \frac{1 + 3h + 3h^2 + h^3 - 1}{h}$$
$$= \lim_{h \to 0} \frac{3h + 3h^2 + h^3}{h} = \lim_{h \to 0} 3 + 3h + h^2 = 3.$$

(b) Derivative of $g(x) = \sqrt{x}$ at $a = 9$.

Solution:

$$g'(9) = \lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h} = \lim_{h \to 0} \frac{9 + h - 9}{h(\sqrt{9+h} + 3)}$$
$$= \lim_{h \to 0} \frac{h}{h(\sqrt{9+h} + 3)} = \lim_{h \to 0} \frac{1}{\sqrt{9+h} + 3} = \frac{1}{3+3} = \frac{1}{6}.$$

Problem 4. You may use any methods we have learned in class to solve these problems, but show enough work to justify your answers.

(a) If $f(x) = \frac{x^2 + \sqrt{x}}{x^4}$ find $f'(1)$.

Solution:

$$f'(x) = \frac{(2x + 1/2x^{-1/2})x^4 - 4x^3(x^2 + \sqrt{x})}{x^8}$$
$$f'(1) = \frac{(2 + 1/2)1 - 4(1+1)}{1} = -11/2.$$

Alternatively,

$$f'(x) = \frac{d}{dx} x^{-2} + x^{-7/2} = -2x^{-3} - 7/2x^{-9/2}$$
$$f'(1) == -2 - 7/2 = -11/2.$$

(b) Find $\frac{d^2 g}{dx^2}$ if $g(x) = x \sec(x)$.

Solution:

$$g'(x) = \sec(x) + x \sec(x) \tan(x)$$
$$g''(x) = \sec(x) \tan(x) + \sec(x) \tan(x) + x \sec(x) \tan^2(x) + x \sec^3(x).$$

(c) Find an equation of the line tangent to $y = \sin(x)\cos(x)$ at the point $(\pi/3, \sqrt{3}/4)$.

Solution: $y' = \cos(x)\cos(x) - \sin(x)\sin(x) = \cos^2(x) - \sin^2(x)$, so $y'(\pi/3) = 1/4 - 3/4 = -1/2$. Thus the tangent line is

$$y - \sqrt{3}/4 = -1/2(x - \pi/3).$$