2 Spanning sets, linear independence, and bases

We began the course by discussing the spaces $\mathbb{R}^n$, which are "normal" Euclidean space. We defined vector spaces in an attempt to generalize their most important properties.

One of the nicest things about $\mathbb{R}^n$ is the existence of *coordinates*. Rather than, say, just drawing a point on a graph, or perhaps giving an angle and a distance, we can specify a point in $\mathbb{R}^3$ by giving its x-coordinate, its y-coordinate, and its z-coordinate. And similarly, we can specify a point in $\mathbb{R}^7$ by specifying seven real-number coordinates. We would like to find a similar concept we can apply to any vector space V.

We'll see that any "coordinate system" will need to have two basic properties: first, we want it to represent any vector in our vector space; second, we want it to represent each vector only once. We will treat these two criteria separately, and then show that we can always find a set that has both properties, which we will call a "basis".

2.1 Spanning sets

Definition 2.1. If V is a vector space $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a list of vectors in V, then a linear combination of of the vectors in S is a vector of the form

$$\sum_{i=1}^{n} a_i \mathbf{v}_i = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

where $a_i \in \mathbb{R}$ are (real number) scalars.

A linear combination of vectors in V will always itself be an element of V, since V is closed under scalar multiplication and under vector addition.

Geometrically, a linear combination of vectors represents some destination you can reach only going in the directions of your chosen vectors (for any distance. So if I can go north or west, any distance"northwest" will be a linear combination of those vectors. And "southeast" will as well, since we can always go in the "opposite" direction. But "up" will not be.

Remark 2.2. This is a "linear" combination because it combines the vectors in the same way a line or plane does—adding all the vectors together, but with some coefficient. We will revisit this terminology in the next section when we discuss linear functions.

It's totally possible to have a linear combination of infinitely many vectors. But studying these requires some sense of convergence, and thus calculus/analysis. So we won't talk about it in *this* class, except for the occasional aside.

Example 2.3. Let $V = \mathbb{R}^3$ and let $S = \{(1,0,0),(0,1,0)\}$. Then we see that

$$(3,2,0) = 3(1,0,0) + 2(0,1,0) \quad \text{and} \quad (-5,3\pi,0) = -5(1,0,0) + 3\pi(0,1,0)$$

are linear combinations of vectors in S.

However, $(1,1,1)$ is *not* a linear combination of vectors in S. If it were, we would have

$$a(1,0,0) + b(0,1,0) = (1,1,1)$$

and thus $(a,b,0) = (1,1,1)$ which cannot happen for any $a, b \in \mathbb{R}$.

We see that this idea can tell us how coordinates work: a set of coordinates for V is a set of vectors S where we can build any vector in V as a linear combination of vectors in S. So the next natural question is to take a set S and ask what vectors we can get by taking linear combinations of vectors in S.

Definition 2.4. Let V be a vector space $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a set of vectors in V. We say the *span* of S is the set of all linear combinations of vectors in S, and write it $\text{Span}(S)$ or $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$.

For notational consistency, we define the span of the empty set $\text{Span}(\{\})$ to be the trivial vector space $\mathbf{0} = \{\mathbf{0}\}$.

Example 2.5. As before, take $V = \mathbb{R}^3$ and $S = \{(1,0,0),(0,1,0)\}$. Then

$$\text{Span}(S) = \{a(1,0,0) + b(0,1,0)\} = \{(a,b,0)\}.$$

Now let $T = \{(3,2,0),(13,7,0)\}$. Then

$$\text{Span}(T) = \{a(3,2,0) + b(13,7,0)\} \{(3a+13b, 2a+7b, 0)\}.$$

Notice that these are actually the same set! The first spanning set "looks" nicer, but it's hard to make this sense of "nice" mathematically precise. We'll do our best but it will take a while.

Example 2.6. Take $V = \mathbb{R}^3$ and let $U = \{(1,0,0),(0,1,0),(1,1,0)\}$. What is $\text{Span}(U)$?

We see that

$$\text{Span}(U) = \{a(1,0,0) + b(0,1,0) + c(1,1,0)\} = \}(a+c, b+c, 0)\}.$$

It's not too hard to see (in this case) that this is the same as the set $\text{Span}(\{(1,0,0),(0,1,0)\}) = \{(x,y,0)\}$. It's clear that any element $(x,y,0)$ can also be written as $(a+c, b+c, 0)$ if we take $a = x, b = y, c = 0$. Conversely, if we have real numbers a, b, c, we can set $x = a+c$ and $y = b+c$ and then $(x,y,0) = (a+c, b+c, 0)$.

We have now described the set $\{(a, b, 0)\}$ as the span of three different sets. Two of these sets have had two elements, and one has had three. It's not too difficult to describe it as the span of a set that's as large as we want: for instance, if we take $S = \{(a, b, 0)\}$ to be the set of all vectors with third coordinate zero, then a little thought will tell us that $\text{Span}(S) = S$. In contrast, it turns out we need to start with at least two elements to span all of S; we will prove this in section 2.2.

We've discussed the idea of spanning algebraically; what is happening geometrically? Recall that each vector gives us a direction and a distance. Since we can multiply our vectors by any scalar, that means we can go any distance in that direction. And since we can add vectors together, that means we can go in one direction, and then another direction. So the span of a set is all of the places I can get to by only going in the direction of vectors in that set.

Let's return to our first example. Our set S included the vectors $(1, 0, 0)$ and $(0, 1, 0)$; those correspond to "north" and "east" in Euclidean threespace. So the span of S is the set of all locations I can get to by going some distance north and then some distance east. But neither of these vectors moves me at all up and down, so I cannot change my height.

In our third example, we had the vectors $(1, 0, 0), (0, 1, 0), (1, 1, 0)$. So we can go north, or east, or north-east. But north-east doesn't open any new options since we could already go north and then east. And we still can't change our height.

Example 2.7. Let $S = \{(1, 2, 3, 4), (1, 1, 1, 1)\}$. Is $(0, 0, 2, 2)$ in $\text{Span}(S)$? Is $(0, 1, 2, 3)$?

$(0, 0, 2, 2)$ is not in $\text{Span}(S)$, since $a(1, 2, 3, 4) + b(1, 1, 1, 1) = (a + b, 2a + b, 3a + b, 4a + b)$ and we can't simultaneously have $a = -b, 2a = -b, 3a = 2 - b$. Becasue $a = -b$ and $2a = -b$ implies $a = 0$ and thus $b = -$. But then $3a + b = 0$ as well, so we can't have $(0, 0, 2, 2)$ as a linear combination of elements of S.

$(0, 1, 2, 3)$ is in $\text{Span}(S)$, since $1(1, 2, 3, 4) + (-1)(1, 1, 1, 1) = (0, 1, 2, 3)$.

Example 2.8. Let $S = \{\sin^2, \cos^2, \tan^2\}$. Is $1 \in \text{Span}(S)$? Is $\sec^2 \in \text{Span}(S)$?

We know from trigonometry that $\sin^2 + \cos^2 = 1$, so $1 \in \text{Span}(S)$. Then we know that $\sin^2 + \cos^2 + \tan^2 = 1 + \tan^2 = \sec^2 \in \text{Span}(S)$.

Spans are really convenient to work with because the span of any set will always be a subspace.

Proposition 2.9. *If V is a vector space and $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\} \subset V$, then $\text{Span}(S)$ is a subspace of V.*

Proof. We know that $S \subset V$, and since any linear combination of vectors in V is itself a vector in V, we know that $\mathrm{Span}(S) \subset V$. So we just need to check the three subspace conditions.

1. We know that $0\mathbf{v} = \mathbf{0}$ for any $\mathbf{v} \in V$. So we have
$$0\mathbf{u}_1 + 0\mathbf{u}_2 + \cdots + 0\mathbf{u}_n = \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0}.$$
Thus $\mathbf{0} \in \mathrm{Span}(S)$.

2. Suppose $\mathbf{v}, \mathbf{w} \in \mathrm{Span}(S)$. This implies that we can write
$$\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{v}_n \qquad \mathbf{w} = b_1\mathbf{w}_1 + \cdots + b_n\mathbf{w}_n$$
for some $a_i, b_i \in \mathbb{R}$. Thus
$$\begin{aligned}\mathbf{v} + \mathbf{w} &= (a_1\mathbf{u}_1 + \cdots + a_n\mathbf{v}_n) + (b_1\mathbf{w}_1 + \cdots + b_n\mathbf{w}_n)\\ &= (a_1 + b_1)\mathbf{u}_1 + \cdots + (a_n + b_n)\mathbf{u}_n \in \mathrm{Span}(S).\end{aligned}$$

3. Suppose $r \in \mathbb{R}$ and $\mathbf{v} \in \mathrm{Span}(S)$. Then we can write
$$\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{v}_n$$
for some $a_i \in \mathbb{R}$. Then
$$r\mathbf{v} = r\left(a_1\mathbf{u}_1 + \cdots + a_n\mathbf{v}_n\right) = (ra_1)\mathbf{u}_1 + \cdots + (ra_n)\mathbf{u}_n \in \mathrm{Span}(S).$$

Thus we see that $\mathrm{Span}(S)$ is a subspace of V. □

So we see that every set spans *some* vector space. In fact, this gives us another way to think of the span of a set.

Corollary 2.10. *If V is a vector space and $S \subseteq V$, then* $\mathrm{Span}(S)$ *is the smallest subspace of V containing S.*

Proof. We just showed in proposition 2.9 that $\mathrm{Span}(S)$ is a subspace of V, and of course it contains S. So we just need to show that there's no smaller subspace. In particular, I'll prove that if W is a subspace of V, and $S \subseteq W$, then $\mathrm{Span}(S) \subseteq W$.

So suppose W is a subspace of V and $S \subseteq W$. Let $\mathbf{v} \in \mathrm{Span}(S)$. The $\mathbf{v}$ is a linear combination of vectors in S. But $S \subseteq W$, so $\mathbf{v}$ is a linear combination of vectors in W, and thus an element of W since W is a vector space. Thus any element of $\mathrm{Span}(S)$ is an element of W, so $\mathrm{Span}(S) \subseteq W$. □

This idea of spanning allows us to generate a set of "coordinates" for a vector space.

Definition 2.11. The set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \subseteq V$ is a *spanning set* for V if every vector in V can be written as a linear combination of vectors in S. That is, S is a spanning set for V if $\mathrm{Span}(S) = V$.

Example 2.12. Which of the following are spanning sets for $\mathbb{R}^3$?

1. $\{(1,0,0),(0,1,0),(0,0,1),(2,1,1)\}$
2. $\{(1,0,0),(0,1,0),(2,1,1)\}$
3. $\{(1,1,1),(1,1,0),(0,1,1)\}$
4. $\{(1,1,1),(1,1,0),(0,0,1)\}$
5. $\{(1,2,3),(3,2,1)\}$

We can check this by seeing which vectors we can make as linear combinations of the vectors in each set. Thus for each set, we want to see if we can find coefficients to make (a,b,c) a linear combination of the given vectors.

1. $$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_4 \\ \alpha_2 + \alpha_4 \\ \alpha_3 + \alpha_4 \end{bmatrix}$$

   thus we need to solve the system of equations

   $$a = \alpha_1 + 2\alpha_4 \qquad b = \alpha_2 + \alpha_4 \qquad c = \alpha_3 + \alpha_4$$

   Later in the course we will talk about systematic ways to approach this problem; but for now we can see that $\alpha_1 = a, \alpha_2 = b, \alpha_3 = c, \alpha_4 = 0$ is a solution to this system. So this is a spanning set for $\mathbb{R}^3$.

2. Again, we need to solve

   $$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_3 \\ \alpha_2 + \alpha_3 \\ \alpha_3 \end{bmatrix}$$

thus we need to solve the system of equations

$$a = \alpha_1 + 2\alpha_3 \qquad b = \alpha_2 + \alpha_3 \qquad c = \alpha_3.$$

This gives us $\alpha_3 = c$, and thus $\alpha_2 = b - \alpha_3 = b - c$ and $\alpha_1 = a - 2\alpha_3 = a - 2c$. We can check this by writing that

$$a - 2c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b - c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a - 2c + 2c \\ b - c + c \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

3. We can use the same approach. We need to solve

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_3 \end{bmatrix}$$

which gives us the three equations

$$a = \alpha_1 + \alpha_2 \qquad b = \alpha_1 + \alpha_2 + \alpha_3 \qquad c = \alpha_1 + \alpha_3$$

We then see that any solution requires $b - a = \alpha_3$, and then $c = \alpha_1 + \alpha_3 = \alpha_1 + b - a$ so $\alpha_1 = a - b + c$; and then we have

$$b = (a - b + c) + \alpha_2 + (b - a) = c + \alpha_2$$

so $\alpha_2 = b - c$. Thus our equations are solved by

$$\alpha_1 = a - b + c \qquad \alpha_2 = b - c \qquad \alpha_3 = b - a$$

and this is a spanning set for $\mathbb{R}^3$.

4. This is very similar to the last problem. We try to solve

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_3 \end{bmatrix}$$

which gives us the three equations

$$a = \alpha_1 + \alpha_2 \qquad b = \alpha_1 + \alpha_2 \qquad c = \alpha_1 + \alpha_3.$$

We immediately see that $a = \alpha_1 + \alpha_2 = b$, so we can't get any vectors with $a \neq b$, so this is not a spanning set.

Taking it a bit further, we see that we can choose a to be anything we like, and c to be anything we like (e.g. set $\alpha_1 = 0, \alpha_2 = a, \alpha_3 = c$). Thus

$$\operatorname{Span}(S) = \{(a, b, c) : a = b\}$$

which is a plane.

5. We solve

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3\alpha_1 + \alpha_2 \\ 2\alpha_1 + 2\alpha_2 \\ \alpha_1 + 3\alpha_2 \end{bmatrix}$$

which gives us the three equations

$$a = 3\alpha_1 + \alpha_2 \qquad b = 2\alpha_1 + 2\alpha_2 \qquad c = \alpha_1 + 3\alpha_2.$$

Solving this we have

$$\begin{aligned} \alpha_2 &= a - 3\alpha_1 \\ \alpha_1 &= c - 3\alpha_2 = c - 3(a - 3\alpha_1) = c - 3a + 9\alpha_1 \\ \alpha_1 &= \frac{3a - c}{8} \\ \alpha_2 &= a - 3\frac{3a - c}{8} = a - \frac{9}{8}a + \frac{3}{8}c \\ &= \frac{3c - a}{8} \end{aligned}$$

Which means that α_1, α_2 are completely determined by a and c without any reference to b; in fact we see that

$$b = 2\alpha_1 + 2\alpha_2 = \frac{3c - a}{4} + \frac{3a - c}{4} = \frac{a + c}{2}.$$

Thus $\operatorname{Span}(S) = \{(a, b, c) : b = (a + c)/2\}$.

Note that the failure of this set to span is not surprising, since $\mathbb{R}^3$ is "three-dimensional" and we only started with two possible directions to go.

Example 2.13. Which of the following are spanning sets for $\mathcal{P}_3(x)$?

1. $\{1, x, x^2, x^3\}$

   We need to see if we can write an arbitrary polynomial as a linear combination of these elements. So we write

   $$a_0 + a_1x + a_2x^2 + a_3x^3 = \alpha_0 \cdot 1 + \alpha_1 \cdot x + \alpha_2 \cdot x^2 + \alpha_3 \cdot x^3$$

   which gives us the linear equations

   $$a_0 = \alpha_0 \qquad a_1 = \alpha_1 \qquad a_2 = \alpha_2 \qquad a_3 = \alpha_3$$

   which...come presolved. So there is a solution to this system, and $\mathrm{Span}(\{1, x, x^2, x^3\}) = \mathcal{P}_3(x)$.

2. $\{1 + x, x^2 + x^3, x + x^2, 1 + x^3\}$

   As before, we try to write a generic polynomial as a linear combination here. We write

   $$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 &= \alpha_0(1 + x) + \alpha_1(x^2 + x^3) + \alpha_2(x + x^2) + \alpha_3(1 + x^3) \\ &= (\alpha_0 + \alpha_3) + (\alpha_0 + \alpha_2)x + (\alpha_1 + \alpha_2)x^2 + (\alpha_1 + \alpha_3)x^3 \end{aligned}$$

   which gives us the system of equations

   $$a_0 = \alpha_0 + \alpha_3 \qquad a_1 = \alpha_0 + \alpha_2 \qquad a_2 = \alpha_1 + \alpha_2 \qquad a_3 = \alpha_1 + \alpha_3$$

   and solving gives

   $$\begin{aligned} \alpha_0 &= a_0 - \alpha_3 \\ \alpha_3 &= a_3 - \alpha_1 \\ \alpha_1 &= a_2 - \alpha_2 \\ \alpha_2 &= a_1 - \alpha_0 \\ \alpha_0 &= a_0 - a_3 + a_2 - a_1 + \alpha_0 \\ 0 &= a_0 - a_3 + a_2 - a_1 \\ a_0 + a_2 &= a_1 + a_3 \end{aligned}$$

   so we see we have a constraint and this is *not* a spanning set for $\mathcal{P}_3(x)$.

   In fact we see that the span is the set of all polynomials where the sum of the even-degree coefficients is the same as the sum of the odd-degree coefficients. And we can go back and check that this is a property that all of our original vectors have, and that is stable under addition and scalar multiplication.

Obviously answering this question effectively requires a thorough study of solving systems of equations like this; we will return to this question in great detail in the future.

We finish with a few facts about spans and spanning sets:

Proposition 2.14. *Let V be a vector space and $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. Then*

1. $\mathbf{0} \in \operatorname{Span}(S)$.
2. *If $S \subseteq T$ then* $\operatorname{Span}(S) \subseteq \operatorname{Span}(T)$.
3. *If $\mathbf{u} \in \operatorname{Span}(S)$, then we write $S \cup \{\mathbf{u}\} = \{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{u}\}$ for the set containing everything in S, and also the element $\mathbf{u}$. Then*
$$\operatorname{Span}(S) = \operatorname{Span}(S \cup \{\mathbf{u}\}).$$
4. *If W is a subspace of V then* $\operatorname{Span}(W) = W$.

Proof.

1. $0 = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n \in \operatorname{Span}(S)$.
2. Set $T = \{\mathbf{v}_1, \dots, bvecv_n, \mathbf{u}_1, \dots, \mathbf{u}_m\}$. Suppose $\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \in \operatorname{Span}(S)$. Then
$$\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n + 0\mathbf{u}_1 + \dots 0\mathbf{u}_m \in \operatorname{Span}(T).$$
3. To prove two sets are equal, it's generally easiest to prove each one is a subset of the other—that is, for each set, we take an arbitrary element of that set and prove it is also an element of the other set.

   We know that $S \subseteq S \cup \{\mathbf{u}\}$, so by part (2) we know that $\operatorname{Span}(S) \subseteq \operatorname{Span}(S \cup \{\mathbf{u}\})$.

   So suppose $\mathbf{w} = b_1\mathbf{u}_1 + \cdots + b_n\mathbf{v}_n + c\mathbf{u} \in \operatorname{Span}(S)$. We know $\mathbf{u} \in \operatorname{Span}(S)$ so we can write $\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$. Then
$$\begin{aligned} \mathbf{w} &= b_1\mathbf{u}_1 + \cdots + b_n\mathbf{v}_n + c\mathbf{u} \\ &= b_1\mathbf{u}_1 + \cdots + b_n\mathbf{v}_n + c\,(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) \\ &= (b_1 + ca_1)\mathbf{u}_1 + \cdots + (b_n + ca_n)\mathbf{u}_n \in \operatorname{Span}(S). \end{aligned}$$

   Thus $\operatorname{Span}(S) \subseteq \operatorname{Span}(S \cup \{\mathbf{u}\})$ and $\operatorname{Span}(S \cup \{\mathbf{u}\}) \subseteq \operatorname{Span}(S)$, so we know that $\operatorname{Span}(S) = \operatorname{Span}(S \cup \{\mathbf{u}\})$.
4. We know that $W \subseteq \operatorname{Span}(W)$, so we just need to show that $\operatorname{Span}(W) \subseteq W$. Let $\mathbf{w} = a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n \in \operatorname{Span}(W)$, where $\mathbf{w}_1, \dots, \mathbf{w}_n \in W$. Then since W is a vector space, it is closed under linear combinations, so any linear combination of $\mathbf{w}_1, \dots, \mathbf{w}_n$ is in W. Thus in particular $\mathbf{w} \in W$.

□

2.2 Linear Independence

This idea of spanning sets answers half of our original question. If we have a spanning set for V, we can write our vectors as sums of elements of the spanning set. But recall we also want this representation to be *unique*—we want to know that if we give two different sets of "coordinates" that they actually represent distinct vectors.

In the previous section, we wanted to study the span of a set of vectors—which tells you how many places you can get with them. Now we want to measure the redundancy: Do we have more vectors in our spanning set than we need?

Definition 2.15. We say a set of vectors $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subseteq V$ is *linearly independent* if the only scalars solving the equation

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$$

are $a_1 = \cdots = a_n = 0$.

If a set of vectors is not linearly independent, we call it *linearly dependent* and there is a *linear dependence* relationship among the vectors.

Example 2.16.

1. The set $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ is linearly independent: suppose

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}.$$

Then we have the system of equations $a = 0, b = 0, c = 0$ and thus all the scalars are zero.

2. The set $S = \{(1,0,0),(0,1,0)\}$ is linearly independent. Suppose

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}a\\b\\0\end{bmatrix}.$$

Then we have the system of equations $a = 0, b = 0$ and thus all the scalars are zero.

3. The set S $= \{(1,0,0),(0,1,0),(1,1,0)\}$ is not linearly independent, since

$$1\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} + (-1)\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = \mathbf{0}.$$

4. Any set containing the zero vector is linearly dependent, since $1 \cdot \mathbf{0} = \mathbf{0}$ but $1 \neq 0$.

There are a few ways we can think about linear independence. One is that a linearly independent set is one where the zero vector can be expressed uniquely—$\mathbf{0}$ is in the span of any set, but it is only in the span of a linearly independent set in one way. In fact, this is enough to make *every* vector expressed uniquely.

Proposition 2.17. *Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a subset of a vector space V. Then S is linearly independent if and only if every vector in* $\operatorname{Span}(S)$ *can be expressed uniquely as a linear combination of vectors in S.*

Proof. Suppose S is linearly dependent. Then by definition of linear independence there are $a_i \in \mathbb{R}$ such that

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n$$

and thus the expression of $\mathbf{0}$ as a linear combination of vectors in S is not unique.

Now suppose not every vector in $\operatorname{Span}(S)$ can be expressed uniquely as a linear combination of vectors in S. By definition of span, every vector in $\operatorname{Span}(S)$ can be represented as a linear combination of vectors in S, so it must be the case that some vector is not represented uniquely, and thus can be written as a linear combination of elements of S in two different ways.

Suppose $\mathbf{u}$ is such an element. Then we have

$$\begin{aligned} a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n &= \mathbf{u} = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n \\ (a_1 - b_1)\mathbf{v}_1 + \cdots + (a_n - b_n)\mathbf{v}_n &= \mathbf{u} - \mathbf{u} = \mathbf{0}. \end{aligned}$$

Thus we can write $\mathbf{0}$ as a nontrivial linear combination of elements of S, so S is linearly dependent. □

Another way to think of this is that in a linearly dependent set, we can express one vector as a linear combination of the others, and thus at least one vector in the set is redundant.

This gives us a geometric interpretation as well. Generally, any one vector defines a line containing it and the origin. Two vectors in general define a plane, three vectors a threespace, and so on. A set is linearly independent if the linear space it defines is as big as you would expect. A set is linearly dependent if the set is smaller—if, say, you have points but they're all on the same line through the origin, so you don't actually get a whole plane.

Lemma 2.18. *A set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly dependent if and only if some element can be written as a linear combination of the others.*

Proof. Without loss of generality, assume $\mathbf{v}_1$ can be written as a linear combination of $\mathbf{v}_2, \dots, \mathbf{v}_n$. (That is, we're assuming one of the vectors can be written as a linear combination of the others, and since order doesn't matter we can assume that it's $\mathbf{v}_1$ to keep the notation simple). Then we have

$$\begin{aligned}
\mathbf{v}_1 &= a_2\mathbf{v}_2 + \dots a_n\mathbf{v}_n \\
\mathbf{v}_1 - \mathbf{v}_1 &= a_2\mathbf{v}_2 + \dots a_n\mathbf{v}_n - \mathbf{v}_1 \\
\mathbf{0} &= (-1)\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.
\end{aligned}$$

Then we have written $\mathbf{0}$ as a nontrivial linear combination of elements of S, and thus S is linearly dependent.

Conversely, suppose S is linearly dependent. Then there are a_i not all zero such that

$$\mathbf{0} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n.$$

We know not all the a_i are zero, so assume without loss of generality that $a_1 \neq 0$. Then we have

$$\begin{aligned}
-a_1\mathbf{v}_1 &= a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \\
\mathbf{v}_1 &= \frac{-a_2}{a_1}\mathbf{v}_2 + \cdots + \frac{-a_n}{a_1}\mathbf{v}_n
\end{aligned}$$

and thus we can write $\mathbf{v}_1$ as a linear combination of the other vectors in S. □

Corollary 2.19. $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ *is linearly dependent if and only if there is some* $\mathbf{v}_i$ *such that* $\operatorname{Span}(S) = \operatorname{Span}(\{\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n\}$.

In practice this is how we prefer to test for linear independence: we try to write one vector as a linear combination of the others. Sometimes this is easy and we're done. Other times this is difficult, or we become convinced it's not possible, and then we have to go back to solving linear equations.

Example 2.20.

1. Let $S = \{(1,1,1), (1,1,0), (0,0,1)\}$. We see that $(1,1,1) = (1,1,0) + (0,0,1)$ so this set is linearly dependent.

2. Let $S = \{(1,1,1), (1,1,0), (0,1,1)\}$. It might look like this is similar, and we could write $(1,1,1)$ somehow as a combination of the other two. But we see that's not actually possible. In fact we write

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ a+b+c \\ b+c \end{bmatrix}$$

and this gives us the system

$$0 = a + b \qquad 0 = a + b + c \qquad 0 = b + c$$

. Subtracting the first from the second gives $c = 0$; plugging this into the third gives $b = 0$ and thus $a = 0$. So these three vectors are linearly independent.

3. Let $S = \{(1, 1, 1), (1, 1, 0), (2, 3, 1), (0, 1, 1)\}$. We cannot write $(1, 1, 1)$ as a linear combination of the other vectors:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a + 2b \\ a + 3b + c \\ b + c \end{bmatrix}$$

gives $a + 2b = 1, a + 3b + c = 1, b + c = 1$. Solving this gives that $a = 1 - 2b$, and thus that $1 - 2b + 3b + c = 1$ so $b + c = 0$. But $b + c = 1$, so there is no solution. (Alternatively, we could notice that the middle coefficient is the sum of the other two, so we can never get $(1, 1, 1)$.

We can't write the first vector as a linear combination of the others. But this doesn't mean that the vectors are linearly independent? Corollary 2.19 says that the vectors are independent if and only if *some* vector is a lienar combination of the others. And we see that

$$2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

and thus set S is linearly dependent.

Proposition 2.21. *If $S \subseteq T$ and T is linearly independent, then S is also linearly independent.*

Proof. Suppose $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq \{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{u}_1, \ldots, \mathbf{u}_m\} = T$, and T is linearly independent. Now suppose their are scalars a_i such that

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}.$$

Then we have

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n + 0\mathbf{u}_1 + \cdots + 0\mathbf{u}_m = 0$$

and since T is linearly independent, we have $a_i = 0$ for every a_i. Thus we see that S is linearly independent.

□

2.3 Vector Space Bases

Having now discussed the two properties we want a coordinate system to have, we can define exactly what we mean by a coordinate system.

Definition 2.22. If V is a vector space and S is a spanning set for V that is also linearly independent, we say that S is a *basis* for V.

Example 2.23. The set $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis for $\mathbb{R}^3$, as we have seen before. We call this set the *standard basis* for $\mathbb{R}^3$, and we write the three elements $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

We can generalize this to $\mathbb{R}^n$. We define the *standard basis vectors* for $\mathbb{R}^n$ by

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \dots \qquad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and the set of standard basis vectors is the *standard basis*. You can check that the standard basis is in fact a basis.

Example 2.24. Every (non-trivial) vector space has more than one basis. The set $S = \{(1,0,0),(1,1,0),(1,1,1)\}$ is a basis for $\mathbb{R}^3$:

First we show that it is a spanning set. Let $(a,b,c) \in \mathbb{R}^3$. Then we want to solve

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_2 + \alpha_3 \\ \alpha_3 \end{bmatrix}$$
$$a = \alpha_1 + \alpha_2 + \alpha_3 \quad b = \alpha_2 + \alpha_3 \quad c = \alpha_3$$

and solving this system gives $\alpha_3 = c, \alpha_2 = b - c, \alpha_1 = a - b$. Thus the set spans.

Now we want to prove linear indepencence. So suppose

$$\mathbf{0} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_2 + \alpha_3 \\ \alpha_3 \end{bmatrix}$$
$$0 = \alpha_1 + \alpha_2 + \alpha_3 \quad 0 = \alpha_2 + \alpha_3 \quad 0 = \alpha_3$$

Then we have $\alpha_3 = 0, \alpha_2 = 0 - \alpha_3 = 0, \alpha_1 = 0 - \alpha_2 - \alpha_3 = 0$. Thus S is linear independent, and thus a basis.

Example 2.25. The set $S = \{(1,0,0),(0,1,0)\}$ is not a basis for $\mathbb{R}^3$. It is linearly independent (since it is a subset of the standard basis, which is linear independent), but it is not a spanning set, since $(0,0,1)$ is not in the span of S.

Example 2.26. The set $S = \{(2,3),(3,4),(4,4)\}$ is a spanning set for $\mathbb{R}^2$ but not a basis. To see that it's a spanning set we solve

$$\begin{bmatrix} a \\ b \end{bmatrix} = \alpha_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \alpha_3 \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2\alpha_1 + 3\alpha_2 + 4\alpha_3 \\ 3\alpha_1 + 4\alpha_2 + 4\alpha_3 \end{bmatrix}$$

giving the system of equations

$$a = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 \qquad\qquad b = 3\alpha_1 + 4\alpha_2 + 4\alpha_3$$

which can be solved by setting

$$\begin{aligned} \alpha_1 &= 3/2\alpha_2 + 2\alpha_3 - a/2 \\ b &= 3(3/2\alpha_2 + 2\alpha_3 - a/2) + 4\alpha_2 + 4\alpha_3 \\ &= 17/2\alpha_2 + 10\alpha_3 - 3a/2 \\ \alpha_3 &= b/10 + 3a/20 - 17\alpha_2/20 \end{aligned}$$

so we can pick α_2 to be anything, and then get values of α_3 and α_1 that solve the system. Thus the set S is a spanning set for $\mathbb{R}^2$.

But we see that S is not linearly independent, since $(2,3) + (1/4)(4,4) = (3,4)$.

Determining whether a set is a basis is sometimes annoying, but doesn't involve anything we haven't already done: a basis is just a set that both spans and is linearly independent, and we can check both properties individually. But we'd like to make things a little simpler.

Further, we want to talk about how "big" a space is, and this should plausibly be determined by how many elements there are in the basis. But since every space has more than one basis, talking about the size of "the" basis is potentially problematic. Fortunately, this is not an actual problem, as we shall see.

Lemma 2.27. *If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ spans a vector space V, and $T = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is a collection of vectors in V with $m > n$, then T is linearly dependent.*

Proof. There are two possible ways to prove this. One involves simply writing out a bunch of linear equations and solving them; however, proving this for any vector space requires more theory of linear equations that we have developed so far. We'll prove it in a more formal and abstract way.

We will start with the set S, and one by one we will trade out vectors in S for vectors in T, and show that we always still have a spanning set. We will suppose T is linearly independent, and show that $m \leq n$.

Since S is a spanning set, we know that $\mathbf{u}_1 \in \operatorname{Span}(S)$, and thus $\{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{u}_1\}$ is linearly dependent by lemma 2.18. Then we can rewrite our linear dependence equation to express $\mathbf{v}_1$ (without loss of generality) as a linear combination of $\{\mathbf{u}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = S_1$, and thus

$$\operatorname{Span}(S) = \operatorname{Span}(\{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{u}_1\}) = \operatorname{Span}(S_1).$$

We can repeat this process: at every step we add the next vector from T to get the set $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_k, \dots, \mathbf{v}_n\}$. Since S_{k-1} is a spanning set, this set is linearly dependent; since the $\mathbf{u}_i$ are linearly independent by hypothesis, we can remove one of the $\mathbf{v}_i$, and without loss of generality we can remove $\mathbf{v}_k$, to obtain the set $S_k = \{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$.

If $m > n$, we can continue until we have replaced every $\mathbf{v}_i$. Then we have $S_n = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a spanning set, and thus $\mathbf{u}_{n+1} \in \operatorname{Span}(S_n)$ and so T is linearly dependent, which contradicts our assumption.

Thus if T is linearly independent, we must have $m \leq n$. Conversely, if $m > n$ then T is linearly dependent, as we stated. □

Corollary 2.28. *$S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $T = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ are two bases for a space V, then they are the same size, i.e. $m = n$.*

Proof. S is a spanning set and T is linearly independent, so we can't have $m > n$ by lemma 2.27. But T is a spanning set and S is linearly independent, so we can't have $n > m$ by lemma 2.27. Thus $n = m$. □

Definition 2.29. Let V be a vector space. If V has a basis consisting of n vectors, we say that V has *dimension* n and write $\dim V = n$. The trivial vector space $\{\mathbf{0}\}$ has dimension 0.

We say that V is *finite–dimensional* if there is a finite set of vectors that spans V. (Thus if V is n-dimensional it is finite-dimensional). Otherwise, we say that V is *infinite–dimensional*.

In this course we will primarily discuss finite dimensional vector spaces.

Example 2.30. The set of standard basis vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$, so $\mathbb{R}^n$ is n-dimensional.

The set $\{1, x, \dots, x^n\}$ is a basis for $\mathcal{P}_n(x)$. This set has $n+1$ vectors, so $\dim \mathcal{P}_n(x) = n+1$.

$\mathcal{P}(x)$ does not have a finite basis. We can see this since the set $S = \{1, x, \dots, x^n\}$ is linearly independent for any n; but every spanning set is at least as big as any linearly independent set, so we can never have a finite spanning set. However, if we allow infinite bases, then $\{1, x, \dots, x^n, \dots\}$ is a basis for $\mathcal{P}(x)$.

Remark 2.31. $\mathcal{C}([a,b], \mathbb{R})$ is infinite–dimensional, but if we allow infinite sums and make convergence arguments it is possible to think of the set $\{1, x, \dots, x^n, \dots\}$ as a sort of ("separable") basis. But this requires analysis and is outside the scope of this course.

The set $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is absurdly huge, and does not have a countable basis. If you believe the axiom of choice it has a basis, as all sets do, but you can't possibly write it down. You can think of it has having "coordinates" given by functions like

$$f_r(x) = \begin{cases} 1 & x = r \\ 0 & x \neq r \end{cases}$$

but this isn't a basis because it would require uncountable sums, which you can't really define.

We'd like to make it easier to check if a set is a basis, and easier to find bases for spaces. We show here that if we start with basically any set, we can turn it into a basis.

Lemma 2.32 (Basis Reduction)**.** *Suppose $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a spanning set for V. Then S can be reduced to a basis for V. That is, there is a subset $B \subseteq S$ that is a basis for V.*

Proof. If S is linearly independent, then it is a basis and we're done.

So suppose S is linearly dependent. Then without loss of generality we can write $\mathbf{v}_n$ as a linear combination of the other vectors in S.

But then $\operatorname{Span}(S) = \operatorname{Span}(\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\})$, and $S_1 = \{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$ is a spanning set for V and a proper subset of S. If S_1 is linearly independent, then it is a basis; if not, we can repeat this process until we reach a linearly independent set, which is our basis B. □

Example 2.33. Let $S = \{(1,1,0), (1,1,1), (0,0,1), (2,7,0)\}$ be a spanning set for $\mathbb{R}^3$. Find a basis $B \subseteq S$ for $\mathbb{R}^3$.

We'll take as given that this is a spanning set, which is not difficult to check. We see that we can write $(1,1,1) = (1,1,0) + (0,0,1)$, so we can remove $(1,1,1)$ without changing the span, and we have $B = \{(1,1,0), (0,0,1), (2,7,0)\} \subseteq S$ is a basis for $\mathbb{R}^3$.

Example 2.34. Let $S = \{(1,2,3), (1,1,1), (5,-2,1), (-4,3,2)\}$ be a spanning set for $\mathbb{R}^3$. Find a basis $B \subseteq S$ for $\mathbb{R}^3$.

We'll take as given that S is a spanning set. We need to write one vector as a linear combination of the others, which is essentially the same problem as finding a nontrivial linear combination equal to zero. So we set up the equation

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}1\\1\\1\end{bmatrix} + c\begin{bmatrix}5\\-2\\1\end{bmatrix} + d\begin{bmatrix}-4\\3\\2\end{bmatrix} = \begin{bmatrix}a+b+5c-4d\\2a+b-2c+3d\\3a+b+c+2d\end{bmatrix}$$

which gives us

$$a+b+5c-4d=0 \qquad 2a+b-2c+3d=0 \qquad 3a+b+c+2d=0.$$

We can compute

$$\begin{aligned}
0 &= a-7c+7d\\
a &= 7c-7d\\
0 &= 2(7c-7d)+b-2c+3d\\
&= b+12c-11d\\
b &= 11d-12c\\
0 &= 3(7c-7d)+(11d-12c)+c+2d\\
&= 10c-8d\\
c &= 4d/5
\end{aligned}$$

which gives

$$\begin{aligned}
c &= 4d/5\\
b &= 11d-12c = 11d-48d/5 = 7d/5\\
a &= 7c-7d = 28d/5-7d = -7d/5\\
0 &= -7\begin{bmatrix}1\\2\\3\end{bmatrix} + 7\begin{bmatrix}1\\1\\1\end{bmatrix} + 4\begin{bmatrix}5\\-2\\1\end{bmatrix} + 5\begin{bmatrix}-4\\3\\2\end{bmatrix}\\
\begin{bmatrix}1\\2\\3\end{bmatrix} &= \begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{4}{7}\begin{bmatrix}5\\-2\\1\end{bmatrix} + \frac{5}{7}\begin{bmatrix}-4\\3\\2\end{bmatrix}.
\end{aligned}$$

Thus $(1,2,3)$ can be written as a linear combination of the other vectors, and so we have $B = \{(1,1,1),(5,-2,1),(-4,3,2)\}$ is a basis for $\mathbb{R}^3$. (We know it is a basis because it is the correct size).

Lemma 2.35 (Basis Padding). *Suppose $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent in V. Then if V has any finite spanning set $T = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$, we can obtain a basis by padding S. That is, there is a basis B for V with $S \subseteq B$.*

Proof. If $T \subset \operatorname{Span}(S)$, then $\operatorname{Span}(T) \subset \operatorname{Span}(S)$, so S is a spanning set for V and thus a basis, so we're done.

So suppose without loss of generality that $\mathbf{u}_1 \notin \operatorname{Span}(S)$. Then $S_1 = \{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{u}_1\}$ is linearly independent by lemma 2.18 since we can't write any element as a linear combination of the others.

If S_1 spans V, then it is a basis and we're done. If not, there is some other $\mathbf{u}_i \notin \operatorname{Span}(S_1)$, so we can repeat the process, and after at most m steps this process will terminate (since we run out of elements in T). When we reach a spanning set, this is our basis.

□

Example 2.36. Let $S = \{(1,1,0),(1,-1,0)\}$. Find a basis $B \supseteq S$ for $\mathbb{R}^3$.

We see that S is linearly independent, so we just need to find a vector that isn't in $\operatorname{Span}(S)$. It's clear that $(0,0,1) \notin \operatorname{Span}(S)$, so we see that $B = \{(1,1,0),(1,-1,0),(0,0,1)\}$ satisfies our requirements.

But there are many choices we could make. It's also the case that $(1,1,1) \notin \operatorname{Span}(S)$, so we see that $B_1 = \{(1,1,0),(1,-1,0),(1,1,1)\}$ also satisfies our requirements.

Example 2.37. Let $T = \{(5,2,-3),(1,-4,7)\}$. Find a basis $B \supseteq T$.

We just need to find a vector that isn't in $\operatorname{Span}(T)$. We can make a guess here and prove it by hand; so for instance it looks like $(1,0,0)$ is not in the span. Indeed, we see that if

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix} + b \begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix} = \begin{bmatrix} 5a+b \\ 2a-4b \\ 7b-3a \end{bmatrix}$$

Then we must have $2a - 4b = 0$ so $a = 2b$, and then from the third coordinate we have $0 = 7b - 3a = 7b - 6b = b$ so $b = 0$ and then $a = 0$ and so the first coordinate must also be zero, a contradiction. Thus $(1,0,0)$ is not in the span of T, and we have that $B = \{(5,2,-3),(1,-4,7),(1,0,0)\} \supseteq T$ is a basis for $\mathbb{R}^3$.

Lemma 2.38. *If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a spanning set for a vector space V, and $T = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is a set with $S \subseteq \operatorname{Span}(T)$, then T is also a spanning set for V.*

Proof. Let $\mathbf{w} \in V$. We want to show that $\mathbf{w}$ is a linear combination of vectors in T. But we know that $\mathbf{w}$ is a linear combination of vectors in S, and that each vector in S is a linear combination of vectors in T. So we can write

$$\begin{aligned}\mathbf{w} &= a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \\ &= a_1(b_{11}\mathbf{u}_1 + \cdots + b_{1m}\mathbf{u}_m) + a_2(b_{21}\mathbf{u}_1 + \cdots + b_{2m}\mathbf{u}_m) \\ &\qquad + \cdots + a_n(b_{n1}\mathbf{u}_1 + \cdots + b_{nm}\mathbf{u}_m) \\ &= (a_1b_{11} + a_2b_{21} + \cdots + a_nb_{n1})\mathbf{u}_1 + \cdots + (a_mb_{1m} + a_2b_{2m} + \cdots + a_nb_{nm})\mathbf{u}_m\end{aligned}$$

as a linear combination of the vectors in T. □

Example 2.39. Suppose we want to show that $T = \{(1,5,2),(2,2,2),(0,3,0)\}$ spans $\mathbb{R}^3$. We just need to find smoe basis that's in the span of T. So we observe

$$\begin{aligned}(0,1,0) &= \frac{1}{3}(0,3,0) \in \operatorname{Span}(T) \\ (1,0,0) &= (2,2,2) + (0,3,0) - (1,5,2) \in \operatorname{Span}(T) \\ (0,0,1) &= \frac{1}{2}(2,2,2) - (1,0,0) - (0,1,0) \in \operatorname{Span}(T)\end{aligned}$$

and thus T contains the standard basis for $\mathbb{R}^3$, and so is a spanning set.

We can push this farther and in more detail, but solving the equations gets more and more annoying. So it seems like a good idea to figure out how to solve systems of linear equations. We will turn our attention to this problem in the next section.