6 Geometry, Angles, and Inner Products

There are two major ideas from vectors in $\mathbb{R}^3$ that we have not yet discussed: the idea of "angle." In $\mathbb{R}^3$ we say that a vector has a magnitude and a direction. And in $\mathbb{R}^3$ we have a pretty good sense of what those mean.

In contrast, consider the space of continuous functions. It's reasonably clear that $\sin(x)$ and x^3+1 do not point in the same direction in the vector space of functions, we can't say whether they're closer to being in the same direction than, say, e^x and e^{2x} are. And we have no idea how to define the magnitude of any of those things.

In this section we want to fix these problems, and understand how magnitude and angles work in vector spaces. We'll start by understanding how this works in $\mathbb{R}^n$.

6.1 The Dot Product

Definition 6.1. Let $\mathbf{u} = (u_1, \dots, u_n), \mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$. We define the *dot product* of $\mathbf{u}$ and $\mathbf{v}$ by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n = \sum_{i=1}^n u_i v_i.$$

This is sometimes also called the *scalar product* on $\mathbb{R}^n$.

Remark 6.2. If we think of $\mathbf{u}$ and $\mathbf{v}$ as $n \times 1$ matrices, we can think of $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$, the product of a $n \times 1$ matrix with a $1 \times n$ matrix.

The dot product has a number of useful properties. First of all, it allows us to define the length or magnitude of a vector.

Definition 6.3. Let $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$. We define the *magnitude* of $\mathbf{v}$ to be

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

Notice that this is just the usual definition of distance; in the plane this is

$$\|(x, y)\| = \sqrt{x^2 + y^2},$$

which is just the pythagorean theorem.

Sometimes it's useful to talk about the distance between two points, rather than the length of a vector. But the distance between two points is the length of the vector between them, so we can define the distance between $\mathbf{x}$ and $\mathbf{y}$ to be

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

The dot product has a few important properties:

Proposition 6.4. *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. *Then:*

1. *(Positive definite)* $\mathbf{u} \cdot \mathbf{u} \geq 0$, *and if* $\mathbf{u} \cdot \mathbf{u} = 0$ *then* $\mathbf{u} = \mathbf{0}$.
2. *(Symmetric)* $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
3. *(Bilinear) The function defined by* $L(\mathbf{x}) = \mathbf{x} \cdot \mathbf{v}$ *is linear, and the function defined by* $T(\mathbf{y}) = \mathbf{u} \cdot \mathbf{y}$ *is linear.*

Proof.

1. $\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + \cdots u_n^2$. Each term is non-negative since each term is a real square, so the sum is non-negative. The sum is zero if and only if each term is zero, if and only if $\mathbf{u} = (0, \ldots, 0) = \mathbf{0}$.
2. $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n = v_1 u_1 + \cdots + v_n u_n = \mathbf{v} \cdot \mathbf{u}$.
3. We'll prove linearity in the first coordinate; the proof for the second coordinate is identical.

   Fix $\mathbf{v} \in \mathbb{R}^n$ and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $r \in \mathbb{R}$. Define $L(\mathbf{x}) = \mathbf{x} \cdot \mathbf{v}$. Then

$$\begin{aligned} L(r\mathbf{x}) &= (r\mathbf{x}) \cdot \mathbf{v} = (rx_1)v_1 + \cdots + (rx_n)v_n = r(x_1 v_1 + \cdots + x_n v_n) = rL(\mathbf{x}) \\ L(\mathbf{x} + \mathbf{y}) &= (\mathbf{x} + \mathbf{y}) \cdot \mathbf{v} = (x_1 + y_1)v_1 + \cdots + (x_n + y_n)v_n \\ &= (x_1 v_1 + \cdots + x_n v_n) + (y_1 v_1 + \cdots + y_n v_n) = L(\mathbf{x}) + L(\mathbf{y}). \end{aligned}$$

□

The dot product also allows us to compute the angle between two vectors.

Proposition 6.5. *If* $\mathbf{u}, \mathbf{v}$ *are two nonzero vectors in* $\mathbb{R}^n$, *and the angle between them is* θ, *then*

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

Proof. We can form a triangle with sides $\mathbf{u}, \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$. Then by the law of cosines (which I'm sure you all remember from high school trigonometry), we have

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

Then we compute

$$\begin{aligned}
\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta &= \frac{1}{2}\left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - \|\mathbf{y}-\mathbf{x}\|^2\right) \\
&= \frac{1}{2}\left(\mathbf{x}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} - (\mathbf{y}-\mathbf{x})\cdot(\mathbf{y}-\mathbf{x})\right) \\
&= \frac{1}{2}\left(\mathbf{x}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} - (\mathbf{y}\cdot\mathbf{y} - \mathbf{x}\cdot\mathbf{y} - \mathbf{y}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{x})\right) \\
&= \frac{1}{2}\left(\mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{x}\right) = \mathbf{x}\cdot\mathbf{y}.
\end{aligned}$$

□

Thus the angle between two vectors is given by $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$.

Example 6.6. Let $\mathbf{u} = (3,4)$ and $\mathbf{v} = (-1,7)$. Then $\mathbf{u}\cdot\mathbf{v} = 3\cdot(-1) + 4\cdot 7 = 25$.

We can compute $\|u\| = \sqrt{3^2+4^2} = 25$ and $\|\mathbf{v}\| = \sqrt{(-1)^2+7^2} = 5\sqrt{2}$. The distance between them is $\|\mathbf{u}-\mathbf{v}\| = \|(4,-3)\| = \sqrt{4^2+(-3)^2} = 5$.

The angle between them is given by

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{25}{5\cdot 5\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\
\theta &= \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.
\end{aligned}$$

We sometimes want to be able to talk about the direction of a vector without worrying about the magnitude. In this case we may wish to comput the *unit vector* given by $\frac{\mathbf{u}}{\|\mathbf{u}\|}$. This vector will clearly have magnitude 1, and point in the same direction that $\mathbf{u}$ does.

If $\mathbf{x},\mathbf{y}$ are unit vectors, then $\cos\theta = \mathbf{x}\cdot\mathbf{y}$.

Example 6.7. The unit vector of $\mathbf{u} = (3,4)$ is $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{5}(3,4) = (3/5, 4/5)$. The unit vector of $\mathbf{v} = (-1,7)$ is $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{5\sqrt{2}}(-1,7) = \left(\frac{-1}{5\sqrt{2}}, \frac{7}{5\sqrt{2}}\right)$.

Then the angle between them is given by

$$\cos\theta = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} \cdot \begin{bmatrix} -1/5\sqrt{2} \\ 7/5\sqrt{2} \end{bmatrix} = \frac{-3}{25\sqrt{2}} + \frac{28}{25\sqrt{2}} = \frac{1}{\sqrt{2}}$$

as before.

There is one more result that is pretty trivial in the case of $\mathbb{R}^n$, but will be very important when we generalize.

Theorem 6.8 (Cauchy-Schwarz Inequality). *If* $\mathbf{u}, \mathbf{v}$ *are vectors in* $\mathbb{R}^n$*, then*

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|. \tag{2}$$

Furthermore, the two sides are equal if and only if either one of the vectors is $\mathbf{0}$*, or* $\mathbf{u} = r\mathbf{v}$ *for some* $r \in \mathbb{R}$*.*

Proof. Recall that $0 \leq |\cos\theta| \leq 1$, with $|\cos\theta| = 1$ if and only if $\theta = n\pi$ for some integer n. Thus

$$|\mathbf{u} \cdot \mathbf{v}| = |\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta| = \|\mathbf{u}\|\|\mathbf{v}\||\cos\theta| \leq \|\mathbf{u}\|\|\mathbf{v}\|.$$

Further, the equality holds only if $\|\mathbf{u}\| = 0$, $\|\mathbf{v}\| = 0$, or $\cos\theta = 1$. In the third case this means the angle between the two vectors is an integer multiple of π, so they either point in the same direction, or in opposite directions. □

180 degree angles are important, but so are right angles. If two vectors are at a right angle to each other, then we have

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\pi/2 = \|\mathbf{u}\|\|\mathbf{v}\| \cdot 0 = 0.$$

We give a special name to these vectors:

Definition 6.9. We say that $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example 6.10.

1. $\mathbf{0}$ is orthogonal to every vector.
2. $(3, 2)$ and $(-4, 6)$ are orthogonal in $\mathbb{R}^2$.
3. Let $\mathbf{u} = (2, 3, 2)$. Can we find a vector orthogonal to it?

   There are lots of them. (They should form an entire plane, if you think about it). One in particular is $(1, 1, -5/2)$.

The last important idea the dot product gives us is the ability to break a vector up into two components. Given $\mathbf{u}$ and $\mathbf{v}$, we can decompose $\mathbf{u}$ into "the part that points in the direction of $\mathbf{v}$" and "the other part."

Suppose we have two vectors $\mathbf{u}$ and $\mathbf{v}$, with angle θ between them. These form two sides of a triangle, with the third side given by $\mathbf{u} - \mathbf{v}$. But we can also draw a line from the endpoint of $\mathbf{u}$ that is perpendicular to $\mathbf{v}$.

We now have a right trangle. The hypotenuse has length $\|\mathbf{u}\|$, so by definition of cosine the length of the adjacent side is $\|\mathbf{u}\| \cos\theta$. But we know that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos theta$$
$$\mathbf{u} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} = \|\mathbf{u}\| \cos\theta$$

so the length of the adjacent side is $\mathbf{u} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|}$. We sometimes call this number the *scalar projection of* $\mathbf{u}$ *onto* $\mathbf{v}$.

Further, we know the direction that the adjacent side is pointing: it's the same direction as $\mathbf{v}$! So we can find this adjacent side as a vector with the formula

$$\mathbf{p} = \mathbf{u} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}.$$

It is not immediately obvious that this is a vector; but most of the dot products give us scalars, with the final $\mathbf{v}$ giving direction.

Finally, we can write $\mathbf{w} = \mathbf{u} - \mathbf{p}$. We will have that $\mathbf{p} \cdot \mathbf{v} = \|\mathbf{p}\| \|\mathbf{v}\|$ since the two vectors point in the same direction; we will have

$$\begin{aligned} \mathbf{w} \cdot \mathbf{v} &= (\mathbf{u} - \mathbf{p}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v} - \mathbf{p} \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} (\mathbf{v} \cdot \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} = \mathbf{0}. \end{aligned}$$

Thus $\mathbf{w}$ is orthogonal to $\mathbf{v}$. We have written $\mathbf{u} = \mathbf{p} + \mathbf{w}$ so that $\mathbf{w}$ is orthogonal to $\mathbf{v}$, and $\mathbf{p}$ points in the same direction as $\mathbf{v}$.

Definition 6.11. If $\mathbf{u}, \mathbf{v}$ are two vectors in $\mathbb{R}^n$, we define the *projection map onto* $\mathbf{v}$ by

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}.$$

Example 6.12. Let's look back at our earlier vectors $\mathbf{u} = (3, 4)$ and $\mathbf{v} = (-1, 7)$. Then we compute

$$\begin{aligned} \text{proj}_{\mathbf{v}} \mathbf{u} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{25}{50} \begin{bmatrix} -1 \\ 7 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 7/2 \end{bmatrix} \\ \mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u} &= \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 7/2 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 1/2 \end{bmatrix}. \end{aligned}$$

6.2 Lines and Planes

We'd like to take the work we just did and use it to understand lines and planes a bit better.

Let's first look at lines. There are a few ways we can describe a line. One is by giving an algebraic equation like $y = 3x + 2$. This should be familiar to you from high school algebra. But there are two other ways we can think of lines in the context of linear algebra.

First, we can think of a line as having a base point, and then a direction (or slope). Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$; then the line through $\mathbf{u}$ in the direction of $\mathbf{v}$ is the set $\{\mathbf{u} + t\mathbf{v} : t \in \mathbb{R}\}$. This is the set of all the points in the direction $\mathbf{v}$ from the point $\mathbf{u}$. We call this the *parametric equation* or *parametrized form* of the line.

Example 6.13. Consider the line $y = 3x + 2$ in the plane. This line goes through the point $(0, 2)$, and we see that it goes in the direction $(1, 3)$: we can either reason this out from the slope, or plug in 1 for x to see the line also goes through $(1, 5)$. Thus our line is given by the set $\{(0, 2) + t(1, 3)\}$.

What's actually going on here? We see that $\{t\mathbf{v}\}$ is actually the span of $\{\mathbf{v}\}$, so our line is just a one-dimensional subspace translated by the vector $\mathbf{u}$. This makes sense, since a one-dimensional subspace is a line through the origin. So a line is just a one-dimensional subspace, plus a shift.

Now that we realize this, we can view lines through the lens of linear operators. The kernel of a linear operator is always a subspace. But recall that the set of solutions to the equation $A\mathbf{x} = \mathbf{b}$ is always a set $\mathbf{u} + \ker(A)$, where $A\mathbf{u} = \mathbf{b}$—which looks exactly like the equation of our line! Thus every line has an equation of the form $A\mathbf{x} = \mathbf{b}$, where A determines the "slope" and $\mathbf{b}$ is given by taking some base point $\mathbf{u}$ and calculating $A\mathbf{u}$.

But we can go one step further: we now have the equation $A\mathbf{x} = A\mathbf{u}$, so by linearity we can rewrite this as $A(\mathbf{x} - \mathbf{u}) = \mathbf{0}$. This is nicer from a linear algebraic perspective, and also saves us the effort of actually computing $A\mathbf{u}$.

Example 6.14. Consider again the line $y = 3x + 2$. The homogeneous form of this line—which has the same slope but goes through the origin—is $y = 3x$. We want to find a matrix that has precisely this line as its kernel. Thus we want $A \begin{bmatrix} x \\ 3x \end{bmatrix} = \mathbf{0}$. The simplest matrix with this property is $[3, -1]$.

We see that $\ker([3, -1]) = \{(x, y) : y = 3x\}$. We just need to find the shift, which means we need a point on the line, which we can take to be $(0, 2)$. We compute $A \begin{bmatrix} 0 \\ 2 \end{bmatrix} = -2$, so the equation for our line is

$$\begin{bmatrix} 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -2,$$

or alternatively

$$\begin{bmatrix} 3 & -1 \end{bmatrix} \begin{bmatrix} x - 0 \\ y - 2 \end{bmatrix} = \mathbf{0}.$$

Notice that if we multiply this out, we just get $3x-y=-2$, which is our original equation.

What exactly is going on here–what does this matrix get us? Well, notice that this matrix multiplication is essentially computing a dot product: we are taking the dot product of $(3,-1)$ with $(x-x_0, y-y_0)$ and checking when it is equal to zero. Thus we're finding all the vectors starting at the base point that are orthogonal to the vector $(3,-1)$. We call this vector the *normal vector* to the line, and we say equations like this are in *normal form.*

Example 6.15. Let $y = 2x-3, z = 3x+5$ be a line in $\mathbb{R}^3$. Find the equations in parametric form and in normal form for this line.

For parametric form, we can say our base point is $\mathbf{u} = (0,-3,5)$. To find the direction, we see that if we increase x by 1, we increase y by 2 and z by 3, so we have $\mathbf{v} = (1,2,3)$. Thus the parametric form of the line is $\{(0,-3,5) + t(1,2,3)\}$.

To find the normal form, we need an operator whose kernel is $\{t(1,2,3)\}$. An easy way to do this is to write down the matrix

$$A = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Thus we see that our line is perpendicular both to $(-2,1,0)$ and $(-3,0,1)$, and in fact perpendicular to the plane spanned by both of them. We compute

$$A \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

so the normal form of our equation is

$$\begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}.$$

In normal form we have

$$\begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x-0 \\ y+3 \\ z-5 \end{bmatrix} = \mathbf{0}.$$

Again, if we multiply this out, we get our original equation back. In fact, if we rewrite our original system of equations we get

$$\begin{aligned} -2x + y + 0z &= -3 \\ -3x + 0y + z &= 5 \end{aligned}$$

whose matrix equation is precisely the matrix we found.

We can use our linear algebraic theory of lines to compute various facts about lines. In particular, we can project a vector onto a line:

The vector $\mathbf{w} = \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$ is the shortest vector from the endpoint of $\mathbf{u}$ to any point on the line in the direction of $\mathbf{v}$.

Example 6.16. Find the point on the line $y = \frac{x}{3}$ that is closest to the point $(1, 4)$, and thie distance between the point and the line.

The line $y = x/3$ is in the direction of $\mathbf{v} = (3, 1)$. So we want to project $\mathbf{u} = (1, 4)$ onto $\mathbf{v}$:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{(1,4)\cdot(3,1)}{(3,1)\cdot(3,1)} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{7}{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 21/10 \\ 7/10 \end{bmatrix}.$$

Thus the point on the line closest to $(1, 4)$ is $(2.1, .7)$.

To find the distance, we look at $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = (1, 4) - (2.1, .7) = (-1.1, 3.3) = 1.1(-1, 3)$. We want to find the length of this vector, so we compute

$$\left\| \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\| = \sqrt{\begin{bmatrix} -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix}} = \sqrt{1+9} = \sqrt{10}$$

so $\|\mathbf{u} - \text{proj}\mathbf{v}\mathbf{u}\| = 1.1\sqrt{10} = \frac{11\sqrt{10}}{10}$.

Example 6.17. Now let's instead find the distance between $(1, 4)$ and the line $y = x/3 + 2$. Our projection operation doesn't quite work here, since $y = x/3 + 2$ doesn't give us a steady vector.

But notice that none of the geometry changes if we translate. So we will get the same answer if we find the distance between $(1, 2)$ and the line $y = x/3$. As before, the vector in the direction of this line is $(3, 1)$. We project $(1, 2)$ onto the vector $(3, 1)$:

$$\mathrm{Proj}_{(3,1)}(1,2) = \frac{(1,2)\cdot(3,1)}{(3,1)\cdot(3,1)} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{5}{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}.$$

The shortest-path vector from $(1, 2)$ to the line is then given by $(1, 2) - (3/2, 1/2) = (-1/2, 3/2)$, which has length

$$\left\| \begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix} \right\| = \sqrt{1/4 + 9/4} = \frac{\sqrt{10}}{2}.$$

We can take the same logic we applied to lines, and now apply it to planes in $\mathbb{R}^3$. A plane requires two parameters to describe; we can think of a plane as being the set $\{\mathbf{u} + t\mathbf{v} + s\mathbf{w} : t, s \in \mathbb{R}\}$ for some base point $\mathbf{u}$ and directions $\mathbf{v}, \mathbf{w}$.

Example 6.18. Consider the plane $z = 2x - y + 3$. This plane clearly contains the point $(0, 0, 3)$. We see that increasing x by 1 increases z by 2, so one direction vector we can use is $(1, 0, 2)$. Similarly, we can also use $(0, 1, -1)$. Thus our plane is parametrized by $\{(0, 0, 3) + t(1, 0, 2) + s(0, 1, -1)\}$.

Again we can also look at planes as the two-dimensional kernel to some operator, plus a base point. Our operator will then have rank 1 and nullity 2, and so again we're essentially taking a dot product: the plane is all the vectors orthogonal to some *normal vector*.

Example 6.19. So we want to find a vector orthogonal to both $(1, 0, 2)$ and $(0, 1, -1)$. We could find this by computing a cross product, but cross products are not very linear-algebraic (and don't work outside the special case of three dimensions). But we're just looking for a vector (x, y, z) that solves

$$(x, y, z) \cdot (1, 0, 2) = 0 \qquad\qquad (x, y, z) \cdot (0, 1, -1) = 0.$$

Thus in fact we're looking for a vector that solves

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

and so we're just looking for the kernel of the matrix given by our original equations: we want the kernel of

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$$

This is already row-reduced! (And if it weren't, it would be fairly easy to reduce). Thus our kernel is $\{(-2\alpha, \alpha, \alpha)\}$, and a specific solution is $(-2, 1, 1)$. (We should check that this is orthogonal to both the original vectors, which it is).

Thus the normal-form equation of our plane is

$$(-2, 1, 1) \cdot (\mathbf{x} - \mathbf{u}) = \mathbf{0} \qquad \text{or}$$

$$\begin{bmatrix} -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x - 0 \\ y - 0 \\ z - 3 \end{bmatrix} = \mathbf{0}.$$

Notice that if we multiply this out we get back our original equation—so we could have read the normal vector off of the original equation, which is in almost-normal form already.

Example 6.20. Find the equation of the plane passing through the points $(1, 1, 2), (2, 3, 3), (3, -3, 3)$.

We'll take $\mathbf{u} = (1, 1, 2)$ to be our base point. To get two vectors in the plane, we find the vector $\mathbf{v} = (2, 3, 3) - (1, 1, 2) = (1, 2, 1)$ and the vector $\mathbf{w} = (3, -3, 3) - (1, 1, 2) = (2, -4, 1)$. Thus the parametrized form is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

To find the normal form, we need a vector perpendicular to $\mathbf{v}$ and $\mathbf{w}$. We write down the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & -8 & -1 \end{bmatrix} \to \begin{bmatrix} 4 & 0 & 3 \\ 0 & 8 & 1 \end{bmatrix}$$

which has kernel $\{(-3/4\alpha, -1/8\alpha, \alpha)\}$, and specific solution $(-6, -1, 8)$ or $(6, 1, -8)$. Thus the normal form of the equation for our plane is

$$\begin{bmatrix} 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 1 \\ z - 2 \end{bmatrix} = \mathbf{0}.$$

Example 6.21. Find the distance from the point $(2, 0, 0)$ to the plane $x + 2y + 2z = 0$.

We want to draw a perpendicular vector from $(2,0,0)$ to the plane $x+2y+2z=0$. We notice that this vector will just be the projection of $(2,0,0)$ onto the normal vector. Since the equation is already

$$\begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

we know the normal vector is $\mathbf{n} = (1,2,2)$. Thus we just want to compute

$$\begin{aligned}
\operatorname{proj}_{(1,2,2)}(2,0,0) &= \frac{(2,0,0)\cdot(1,2,2)}{(1,2,2)\cdot(1,2,2)} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\
&= \frac{2}{9} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\
|\operatorname{proj}_{1,2,2}(2,0,0)| &= \frac{2}{9} \left\| \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\| = \frac{2}{9}\sqrt{1+4+4} = \frac{2}{3}.
\end{aligned}$$

Thus the distance is 2/3.