Math 214 Spring 2017
Linear Algebra HW 10 Solutions
Due Wednesday, April 12

For all these problems, justify your answers.

1. Let $\mathbf{u} = (2, 1, 3)$ and $\mathbf{v} = (6, 3, 9)$.
   (a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.
   (b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.
   (c) Verify that $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}$.

   Solution:

   (a)

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= 42 \\ \|\mathbf{u}\| &= \sqrt{14} \\ \|\mathbf{v}\| &= 3\sqrt{14} \\ \cos\theta &= \frac{42}{3 \cdot 14} = 1 \\ \theta &= \pi/2. \end{aligned}$$

   (b)

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{42}{126}\mathbf{v} = \frac{1}{3}\begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

   (c) $\operatorname{Proj}_{\mathbf{v}} \mathbf{u} = \mathbf{u}$ so $u - \operatorname{Proj}_{\mathbf{v}} \mathbf{u} = \mathbf{0}$ is orthogonal to everything, including $\mathbf{u}$. In particular, we compute

$$\left(\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right) \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 0.$$

2. Let $\mathbf{u} = (2, -5, 4)$ and $\mathbf{v} = (1, 2, -1)$.
   (a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.
   (b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

(c) Verify that $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ is orthogonal to $\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}$.

Solution:

(a) We compute

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= -12 \\
\|\mathbf{u}\| &= \sqrt{45} \\
\|\mathbf{v}\| &= \sqrt{6} \\
\cos\theta &= \frac{-12}{\sqrt{45}\sqrt{6}} =\approx -.73 \\
\theta &\approx 2.39.
\end{aligned}$$

(b)

$$\begin{aligned}
\text{proj}_{\mathbf{v}}\,\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} \\
&= \frac{-12}{6}\mathbf{v} = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}
\end{aligned}$$

(c) We compute that

$$\begin{aligned}
\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u} &= \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \\
\text{proj}_{\mathbf{v}}\,\mathbf{u} \cdot (\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}) &= \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = 0.
\end{aligned}$$

3. Let $\mathbf{u} = (4, 1)$ and $\mathbf{v} = (3, 2)$.

(a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

(b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

(c) Verify that $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ is orthogonal to $\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}$.

Solution:

(a)

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= 14 \\
\|\mathbf{u}\| &= \sqrt{17} \\
\|\mathbf{v}\| &= \sqrt{13} \\
\cos\theta &= \frac{14}{\sqrt{17 \cdot 13}} \approx .94 \\
\theta &\approx .34.
\end{aligned}$$

(b)

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{14}{13} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

(c) We compute

$$\begin{aligned}
\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \frac{14}{13} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 10/13 \\ -15/13 \end{bmatrix} \\
\operatorname{proj}_{\mathbf{v}} \mathbf{u} \cdot (\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}) &= \frac{14}{13}(30/13 - 30/13) = 0.
\end{aligned}$$

4. Let $\mathbf{u} = (3, 5)$ and $\mathbf{v} = (1, 1)$.

   (a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

   (b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

   (c) Verify that $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}$.

   Solution:

   (a)

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= 8 \\
\|\mathbf{u}\| &= \sqrt{34} \\
\|\mathbf{v}\| &= \sqrt{2} \\
\cos\theta &= \frac{8}{2\sqrt{17}} = \frac{4\sqrt{17}}{17} \approx .97 \\
\theta &= .25.
\end{aligned}$$

   (b)

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{8}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

   (c)

$$\begin{aligned}
\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \begin{bmatrix} 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
\operatorname{proj}_{\mathbf{v}} \mathbf{u}(\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}) &= \begin{bmatrix} 4 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -4 + 4 = 0.
\end{aligned}$$

5. Find the point on the line $y = 2x$ that is closest to the point $(5, 2)$, and the distance between them.

   Solution: We want to project $\mathbf{u} = (5, 2)$ onto the vector $\mathbf{v} = (1, 2)$. So we compute

$$\begin{aligned}
\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \frac{9}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9/5 \\ 18/5 \end{bmatrix} \\
\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \begin{bmatrix} 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 9/5 \\ 18/5 \end{bmatrix} = \begin{bmatrix} 16/5 \\ -8/5 \end{bmatrix} \\
\left\| \begin{bmatrix} 16/5 \\ -8/5 \end{bmatrix} \right\| &= \frac{8}{5} \left\| \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\| = \frac{8\sqrt{5}}{5}.
\end{aligned}$$

So the nearest point is $(9/5, 18/5)$, and the distance is $\frac{8\sqrt{5}}{5}$.

6. Find the distance from the point $(1, 1, 1)$ to the plane $2x + 2y + z = 0$.

   Solution: The normal vector is $\mathbf{n} = (2, 2, 1)$, so we want to project $(1, 1, 1)$ onto $(2, 2, 1)$. We compute

$$\mathrm{Proj}_{\mathbf{n}} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \frac{5}{9} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$
$$\left\| \frac{5}{9} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\| = \frac{5}{3}.$$

   Thus the distance is $5/3$. (The nearest point, incidentally, is $(1, 1, 1) - \frac{5}{9}(2, 2, 1) = (-1/9, -1/9, 4/9)$.

7. Write equations for the lines $2x + y = 5$ and $2x + y = 0$ in parametrized form and in normal form.

   Solution: For $2x + y = 5$ we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x - 0 \\ y - 5 \end{bmatrix} = 0 \qquad \text{or}$$
$$\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

   For $2x + y = 0$ we similarly get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x - 0 \\ y - 0 \end{bmatrix} = 0 \qquad \text{or}$$
$$\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

8. Write equations for the plane passing through $(2, 3, 1), (5, 4, 3), (3, 4, 4)$ in parametrized form and in normal form.

   Solution:

   Our base point $\mathbf{u}$ is $(2, 3, 1)$. We can take our vectors to be $\mathbf{v} = (5, 4, 3) - (2, 3, 1) = (3, 1, 2)$ and $\mathbf{w} = (3, 4, 4) - (2, 3, 1) = (1, 1, 3)$, so the parametric form is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

For the normal form, we need a vector perpendicular to $(3, 1, 2)$ and $(1, 1, 3)$. We can find this using the cross product, or we can use row reduction: our normal vector is in the kernel of

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

so we have $\mathbf{n} = (-1, -2, 1)$. Then the normal form of the equation is

$$\begin{bmatrix} -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x - 0 \\ y - 0 \\ z - 0 \end{bmatrix} = 0.$$

9. Find an algebraic equation for the plane normal to $\mathbf{N} = (-3, 6, 2)$ and passing through $(4, 2, -5)$.

Solution:

The normal form is

$$\begin{bmatrix} -3 & 6 & 2 \end{bmatrix} \begin{bmatrix} x - 4 \\ y - 2 \\ z + 5 \end{bmatrix} = 0.$$

Multiplying this out gives

$$\begin{aligned} -3(x - 4) + 6(y - 2) + 2(z + 5) &= 0 \\ -3x + 6y + 2z &= -10. \end{aligned}$$