

Math 214 Spring 2017
Linear Algebra HW 11 Solutions
Due Friday, April 21

For all these problems, justify your answers.

1. Let $V = \mathcal{P}_n(x)$ and fix real numbers x_0, x_1, \dots, x_n be distinct real numbers. For $f, g \in V$, define

$$\langle f, g \rangle = \sum_{i=0}^n f(x_i)g(x_i).$$

Prove this is an inner product on V .

(Hint: See partial proof from class)

Solution: We need to prove that this has three properties.

- (a) Positive Definite: We showed this in class. If f is a real polynomial, then $\langle f, f \rangle = \sum_{i=0}^n f(x_i)^2 \geq 0$ since it is a sum of non-negative numbers. If $\langle f, f \rangle = 0$ then $f(x_i)^2 = 0$ for each x_i , so f has $n+1$ roots. Since f is a degree n polynomial with $n+1$ roots, f is zero.
- (b) Symmetric: We have

$$\langle f, g \rangle = \sum_{i=0}^n f(x_i)g(x_i) = \sum_{i=0}^n g(x_i)f(x_i) = \langle g, f \rangle.$$

- (c) Bilinear (only need to check in the first term because the function is symmetric): we see that

$$\begin{aligned} \langle \alpha f + \beta g, h \rangle &= \sum_{i=0}^n (\alpha f(x_i) + \beta g(x_i))h(x_i) \\ &= \alpha \sum_{i=0}^n f(x_i)h(x_i) + \beta \sum_{i=0}^n g(x_i)h(x_i) = \alpha \langle f, h \rangle + \beta \langle g, h \rangle. \end{aligned}$$

2. Let w_1, \dots, w_n be positive real numbers. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, define

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i w_i.$$

Prove that this is an inner product on \mathbb{R}^n . (The w_i are called the *weights* of the inner product).

Solution: We again need to check three things.

(a) Positive Definite: We have $\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^n x_i^2 w_i \geq 0$ since $x_i^2 \geq 0$ and $w_i > 0$. Further, if $\sum_{i=1}^n x_i^2 w_i = 0$, then we must have $x_i^2 w_i = 0$ for each i , and since $w_i > 0$ this implies that $x_i^2 = 0$ so $x_i = 0$. Thus $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and is equal to zero if and only if $\mathbf{x} = \mathbf{0}$.

(b) Symmetric:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i w_i = \sum_{i=1}^n y_i x_i w_i = \langle \mathbf{y}, \mathbf{x} \rangle.$$

(c) Bilinear:

$$\begin{aligned} \langle \alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z} \rangle &= \sum_{i=1}^n (\alpha x_i + \beta y_i) z_i w_i \\ &= \alpha \sum_{i=1}^n x_i z_i w_i + \beta \sum_{i=1}^n y_i z_i w_i = \alpha \langle \mathbf{x}, \mathbf{z} \rangle + \beta \langle \mathbf{y}, \mathbf{z} \rangle. \end{aligned}$$

3. Let $V = \mathcal{C}([1, 3], \mathbb{R})$, with the usual inner product. Find $\|1\|$ and $\|x\|$. Find the projection of $1 + x$ onto 1 and x .

Solution:

$$\begin{aligned} \|1\| &= \sqrt{\int_1^3 1^2 dx} = \sqrt{x|_1^3} = \sqrt{2} \\ \|x\| &= \sqrt{\int_1^3 x^2 dx} = \sqrt{x^3/3|_1^3} = \sqrt{26/3} \\ \text{proj}_1 1 + x &= \frac{\langle 1 + x, 1 \rangle}{\langle 1, 1 \rangle} 1 = \frac{1}{2} \int_1^3 1 + x dx(1) \\ &= \frac{1}{2} (x + x^2/2)|_1^3(1) = 3(1) \\ \text{proj}_x 1 + x &= \frac{\langle 1 + x, x \rangle}{\langle x, x \rangle} x = \frac{3}{26} \int_1^3 x + x^2 dx(x) \\ &= \frac{3}{26} (x^2/2 + x^3/3)|_1^3(x) = \frac{3}{26} (4 + 26/3)x = \frac{38}{26}x. \end{aligned}$$

4. Prove the Pythagorean law: if \mathbf{u}, \mathbf{v} are orthogonal, then $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$.

Solution:

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} + \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + 0 + 0 + \langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2. \end{aligned}$$

5. Let \mathbf{u}, \mathbf{v} be vectors in an inner product space V , with $\mathbf{v} \neq 0$. Let $\mathbf{p} = \text{proj}_{\mathbf{v}} \mathbf{u}$.

Prove that $\langle \mathbf{u} - \mathbf{p}, \mathbf{p} \rangle = 0$.

Solution: We calculate

$$\begin{aligned}
 \mathbf{p} &= \text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v} \\
 \langle \mathbf{u} - \mathbf{p}, \mathbf{p} \rangle &= \left\langle \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}, \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v} \right\rangle \\
 &= \left\langle \mathbf{u}, \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v} \right\rangle - \left\langle \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}, \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v} \right\rangle \\
 &= \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \langle \mathbf{u}, \mathbf{v} \rangle - \left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \right)^2 \langle \mathbf{v}, \mathbf{v} \rangle \\
 &= \frac{\langle \mathbf{u}, \mathbf{v} \rangle^2}{\langle \mathbf{v}, \mathbf{v} \rangle} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle^2}{\langle \mathbf{v}, \mathbf{v} \rangle} = 0.
 \end{aligned}$$

6. Let $V = \mathcal{C}([- \pi, \pi], \mathbb{R})$ with the usual inner product. Show that $\{1, \sin(x), \cos(x)\}$ is an orthogonal set. Is it orthonormal?

Solution:

$$\begin{aligned}
 \langle 1, \sin(x) \rangle &= \int_{-\pi}^{\pi} 1 \sin(x) dx = -\cos(x)|_{-\pi}^{\pi} = -\cos(\pi) + \cos(-\pi) = 1 - 1 = 0. \\
 \langle 1, \cos(x) \rangle &= \int_{-\pi}^{\pi} 1 \cos(x) dx = \sin(x)|_{-\pi}^{\pi} = \sin(\pi) - \sin(-\pi) = 0 - 0 = 0. \\
 \langle \sin(x), \cos(x) \rangle &= \int_{-\pi}^{\pi} \sin(x) \cos(x) dx = \frac{1}{2} \sin^2(x)|_{-\pi}^{\pi} = \frac{1}{2} (\sin^2(\pi) - \sin^2(-\pi)) \\
 &= \frac{1}{2} (0 - 0) = 0.
 \end{aligned}$$

We can see this isn't an orthonormal set because e.g. $\langle 1, 1 \rangle = \int_{-\pi}^{\pi} 1 dx = 2\pi \neq 1$.

7. Let $V = \mathbb{R}^4$ with the dot product, and let $U = \text{Span}(\{(5, 3, 1, 0), (2, 4, 3, 5), (1, 1, 1, 1)\})$. Use the Gram-Schmidt process to find an orthonormal basis for U .

Solution: We can pick any element to start with. It would probably be simplest to start with $(1, 1, 1, 1)$, but I'm going to start with $(5, 3, 1, 0)$ here because that's the order I wrote them in. Starting with any one is fine.

$$\mathbf{f}_1 = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{u}_1 = \frac{1}{\sqrt{25 + 9 + 1}} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{35}} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned}
\mathbf{f}_2 &= \mathbf{e}_2 - \text{proj}_{\mathbf{f}_1} \begin{bmatrix} 2 \\ 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 5 \end{bmatrix} - \frac{(5, 3, 1, 0) \cdot (2, 4, 3, 5)}{(5, 3, 1, 0) \cdot (5, 3, 1, 0)} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 4 \\ 3 \\ 5 \end{bmatrix} - \frac{25}{35} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -11/7 \\ 13/7 \\ 16/7 \\ 5 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -11 \\ 13 \\ 16 \\ 35 \end{bmatrix} \\
\mathbf{u}_2 &= \frac{1}{\sqrt{11^2 + 13^2 + 16^2 + 35^2}} \begin{bmatrix} -11 \\ 13 \\ 16 \\ 35 \end{bmatrix} = \frac{1}{\sqrt{1771}} \begin{bmatrix} -11 \\ 13 \\ 16 \\ 35 \end{bmatrix} \\
\mathbf{f}_3 &= \mathbf{e}_3 - \text{proj}_{\mathbf{f}_1} \mathbf{e}_3 - \text{proj}_{\mathbf{f}_2} \mathbf{e}_3 \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{(5, 3, 1, 0) \cdot (1, 1, 1, 1)}{(5, 3, 1, 0) \cdot (5, 3, 1, 0)} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix} - \frac{(-11, 13, 16, 35) \cdot (1, 1, 1, 1)}{(-11, 13, 16, 35) \cdot (-11, 13, 16, 35)} \begin{bmatrix} -11 \\ 13 \\ 16 \\ 35 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{9}{35} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix} - \frac{53}{1771} \begin{bmatrix} -11 \\ 13 \\ 16 \\ 35 \end{bmatrix} = \frac{1}{1265} \begin{bmatrix} 55 \\ -203 \\ 334 \\ -60 \end{bmatrix} \\
\mathbf{u}_3 &= \frac{1}{\sqrt{55^2 + 203^2 + 334^2 + 60^2}} \begin{bmatrix} 55 \\ -203 \\ 334 \\ -60 \end{bmatrix} = \frac{1}{3\sqrt{17710}} \begin{bmatrix} 55 \\ -203 \\ 334 \\ -60 \end{bmatrix}.
\end{aligned}$$

We should check that these are orthogonal; we see that

$$\begin{aligned}
\langle \mathbf{u}_1, \mathbf{u}_2 \rangle &= \frac{1}{\sqrt{35 \cdot 1771}} (5 \cdot (-11) + 3 \cdot 13 + 1 \cdot 16) = \frac{1}{\sqrt{35 \cdot 1771}} (-55 + 39 + 16) = 0 \\
\langle \mathbf{u}_1, \mathbf{u}_3 \rangle &= \frac{1}{3\sqrt{35 \cdot 17710}} (5 \cdot 55 + 3 \cdot (-203) + 1 \cdot 334) \\
&= \frac{1}{3\sqrt{35 \cdot 17710}} (275 - 609 + 334) = 0 \\
\langle \mathbf{u}_2, \mathbf{u}_3 \rangle &= \frac{1}{3\sqrt{1771 \cdot 17710}} ((-11)55 + 13(-203) + 16 \cdot 334 + 35(-60)) \\
&= \frac{1}{3 \cdot 1771\sqrt{10}} (-605 - 2639 + 5344 - 2100) = 0.
\end{aligned}$$

8. Let $V = \mathcal{P}_2(x)$ with the inner product $\langle f, g \rangle = \int_{-1}^1 f(t)g(t) dt$.

Following the Gram-Schmidt process, convert $\{1, x, x^2\}$ into an orthonormal basis.

Solution: We have

$$\mathbf{f}_1 = 1 \qquad \mathbf{u}_1 = \frac{1}{\sqrt{\int_{-1}^1 1 dx}} 1 = \frac{1}{\sqrt{2}}.$$

$$\begin{aligned}
\mathbf{f}_2 &= \mathbf{e}_2 - \text{proj}_{\mathbf{f}_1} \mathbf{e}_2 = x - \frac{\int_{-1}^1 x dx}{\int_{-1}^1 1 dx} x = x - \frac{0}{2} x = x \\
\mathbf{u}_2 &= \frac{1}{\sqrt{\int_{-1}^1 x^2 dx}} x = \frac{1}{\sqrt{2/3}} x = \sqrt{3/2} x \\
\mathbf{f}_3 &= \mathbf{e}_3 - \text{proj}_{\mathbf{f}_1} \mathbf{e}_3 - \text{proj}_{\mathbf{f}_2} \mathbf{e}_3 \\
&= x^2 - \frac{\int_{-1}^1 x^2 dx}{\int_{-1}^1 1 dx} 1 - \frac{\int_{-1}^1 x^3 dx}{\int_{-1}^1 x^2} x \\
&= x^2 - \frac{2/3}{2} 1 - \frac{0}{2/3} x = x^2 - 1/3 \\
\mathbf{u}_3 &= \frac{1}{\sqrt{\int_{-1}^1 (x^2 - 1/3)^2 dx}} (x^2 - 1/3) = \frac{1}{\sqrt{8/45}} (x^2 - 1/3) \\
&= \frac{3\sqrt{5}}{2\sqrt{2}} (x^2 - 1/3) = \frac{\sqrt{5}}{2\sqrt{2}} (3x^2 - 1).
\end{aligned}$$

We should check that our basis is in fact orthogonal and orthonormal. We have

$$\begin{aligned}
\langle \mathbf{u}_1, \mathbf{u}_2 \rangle &= \int_{-1}^1 \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{\sqrt{2}} x dx = \frac{\sqrt{3}}{2} \int_{-1}^1 x dx = 0 \\
\langle \mathbf{u}_1, \mathbf{u}_3 \rangle &= \int_{-1}^1 \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{5}}{2\sqrt{2}} (3x^2 - 1) dx = \frac{\sqrt{5}}{4} \int_{-1}^1 3x^2 - 1 \\
&= \frac{\sqrt{5}}{4} (1 - 1 - 1 + 1) = 0. \\
\langle \mathbf{u}_2, \mathbf{u}_3 \rangle &= \int_{-1}^1 \frac{\sqrt{3}}{\sqrt{2}} x \cdot \frac{\sqrt{5}}{2\sqrt{2}} (3x^2 - 1) dx = \frac{\sqrt{15}}{4} \int_{-1}^1 3x^3 - x dx \\
&= \frac{\sqrt{15}}{4} \left(\frac{3}{4} - \frac{1}{3} - \frac{3}{4} + \frac{1}{3} \right) = 0
\end{aligned}$$

9. Let $V = \mathbb{R}^4$ and let $U = \text{Span}(\{(3, 5, 2, 1), (5, 1, -1, -5)\})$. Find an orthonormal basis for U^\perp .

Solution: We first need to find a basis for U^\perp at all. U^\perp is the kernel of the matrix whose rows are a basis for U . We write

$$A = \begin{bmatrix} 3 & 5 & 2 & 1 \\ 5 & 1 & -1 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 22 & 0 & -7 & -26 \\ 0 & 22 & 13 & 20 \end{bmatrix}$$

which gives us $\{(7, -13, 22, 0), (26, -20, 0, 22)\}$ as a basis for U^\perp . We just need to orthonormalize.

$$\mathbf{f}_1 = \begin{bmatrix} 7 \\ -13 \\ 22 \\ 0 \end{bmatrix} \quad \mathbf{u}_1 = \frac{1}{\sqrt{7^2 + 13^2 + 22^2}} \begin{bmatrix} 7 \\ -13 \\ 22 \\ 0 \end{bmatrix} = \frac{1}{702} \begin{bmatrix} 7 \\ -13 \\ 22 \\ 0 \end{bmatrix}$$

$$\begin{aligned}
\mathbf{f}_2 &= \mathbf{e}_2 - \text{proj}_{\mathbf{f}_1} \mathbf{e}_2 = \begin{bmatrix} 26 \\ -20 \\ 0 \\ 22 \end{bmatrix} - \frac{(26, -20, 0, 22) \cdot (7, -13, 22, 0)}{(7, -13, 22, 0) \cdot (7, -13, 22, 0)} \begin{bmatrix} 7 \\ -13 \\ 22 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 26 \\ -20 \\ 0 \\ 22 \end{bmatrix} - \frac{442}{702} \begin{bmatrix} 7 \\ -13 \\ 22 \\ 0 \end{bmatrix} = \frac{1}{27} \begin{bmatrix} 583 \\ -319 \\ -374 \\ 594 \end{bmatrix}
\end{aligned}$$

We should check that these two vectors are orthogonal to each other, and are in the kernel of A . Fortunately they are, so this is an orthonormal basis for U^\perp .

10. Let $\mathbf{u}_1, \mathbf{u}_2$ form an orthonormal basis for \mathbb{R}^2 , and suppose \mathbf{v} is a unit vector. If $\mathbf{v} \cdot \mathbf{u}_1 = 1/2$, compute $|\mathbf{v} \cdot \mathbf{u}_2|$.

Solution: Let $\mathbf{v} \cdot \mathbf{u}_2 = r$. Then we have $\mathbf{v} = (\mathbf{v} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v} \cdot \mathbf{u}_2)\mathbf{u}_2 = \frac{1}{2}\mathbf{u}_1 + r\mathbf{u}_2$.

Since $\mathbf{u}_1, \mathbf{u}_2$ are orthogonal unit vectors, we know that

$$\begin{aligned}
\|\mathbf{v}\|^2 &= \left\| \frac{1}{2}\mathbf{u}_1 + r\mathbf{u}_2 \right\|^2 = \left\| \frac{1}{2}\mathbf{u}_1 \right\|^2 + \|r\mathbf{u}_2\|^2 \\
1^2 &= \left(\frac{1}{2}\right)^2 + r^2 \\
r^2 &= 1 - \frac{1}{2} \\
|r| &= \sqrt{3/4} = \frac{\sqrt{3}}{2}.
\end{aligned}$$