

Math 214 Spring 2017
Linear Algebra HW 3 Solutions
Due Friday, February 10

For all these problems, justify your answers; do not just write “yes” or “no”.

1. (a) Can you write $(1, 3, 2, 4)$ as a linear combination of $(1, 0, 1, 0)$, $(0, 1, 0, 2)$, and $(0, 1, 1, 1)$?

Solution: We need to solve

$$\begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b+c \\ a+c \\ 2b+c \end{bmatrix}$$

and thus get the system of equations

$$1 = a \qquad 3 = b + c \qquad 2 = a + c \qquad 4 = 2b + c.$$

We have $a = 1$ and thus $c = 2 - a = 1$. Then we have $b = 3 - c = 2$ and $2b = 4 - c = 3$, which is not possible, and thus a contradiction. So no.

- (b) Can you write $(1, 3, 2, 4)$ as a linear combination of $(1, 0, 1, 0)$, $(0, 1, 0, 2)$, and $(1, 1, 1, 1)$?

Solution: We need to solve

$$\begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+c \\ b+c \\ a+c \\ 2b+c \end{bmatrix}$$

and thus get the system of equations

$$1 = a + c \qquad 3 = b + c \qquad 2 = a + c \qquad 4 = 2b + c.$$

The first equation says $1 = a + c$ and the third says $2 = a + c$, which is a contradiction. So no.

- (c) Write $4x + 6x^3 - x^5$ as a linear combination of $x + x^3$, $x^3 + x^5$, and $x + x^5$.

Solution: We can do this by trial and error, or systematically. The most systematic approach is to write the equations

$$4x + 6x^3 - x^5 = a(x + x^3) + b(x^3 + x^5) + c(x + x^5) = (a + c)x + (a + b)x^3 + (b + c)x^5$$

and thus we get

$$4 = a + c \qquad 6 = a + b \qquad -1 = b + c.$$

We have $a = 4 - c$, and thus $b = 6 - a = 6 - 4 + c = c + 2$, and thus $-1 = b + c = 2c + 2$ and thus $c = -3/2$. Then $a = 11/2$ and $b = 1/2$. So we have

$$4x + 6x^2 - x^5 = 11/2(x + x^3) + 1/2(x^3 + x^5) - 3/2(x + x^5).$$

2. Let $V = \mathbb{R}^3$.

(a) Is $S = \{(1, 2, 3), (2, 3, 4), (3, 4, 5)\}$ a spanning set for \mathbb{R}^3 ?

Solution: We try to solve

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 + 3\alpha_3 \\ 2\alpha_1 + 3\alpha_2 + 4\alpha_3 \\ 3\alpha_1 + 4\alpha_2 + 5\alpha_3 \end{bmatrix}$$

and get the system of equations

$$a = \alpha_1 + 2\alpha_2 + 3\alpha_3 \qquad b = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 \qquad c = 3\alpha_1 + 4\alpha_2 + 5\alpha_3.$$

We get

$$\begin{aligned} \alpha_1 &= a - 2\alpha_2 - 3\alpha_3 \\ 3\alpha_2 &= b - 2\alpha_1 - 4\alpha_3 = b - 2(a - 2\alpha_2 - 3\alpha_3) - 4\alpha_3 \\ &= b - 2a + 4\alpha_2 + 2\alpha_3 \\ \alpha_2 &= 2a - b - 2\alpha_3 \\ 5\alpha_3 &= c - 3\alpha_1 - 4\alpha_2 = c - 3(a - 2\alpha_2 - 3\alpha_3) - 4\alpha_2 \\ &= c - 3a + 2\alpha_2 + 9\alpha_3 = c - 3a + 2(2a - b - 2\alpha_3) + 9\alpha_3 \\ &= c + a - 2b + 5\alpha_3 \\ 0 &= c + a - 2b \end{aligned}$$

Thus any vector in the span must have $2b = a + c$, and so the set does not span V .

(b) Is $T = \{(1, 2, 3), (2, 3, 4), (0, 1, 1)\}$ a spanning set for \mathbb{R}^3 ?

Solution: We try to solve

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ 2\alpha_1 + 3\alpha_2 + \alpha_3 \\ 3\alpha_1 + 4\alpha_2 + \alpha_3 \end{bmatrix}$$

and get the system of equations

$$a = \alpha_1 + 2\alpha_2 \qquad b = 2\alpha_1 + 3\alpha_2 + \alpha_3 \qquad c = 3\alpha_1 + 4\alpha_2 + \alpha_3.$$

We get

$$\begin{aligned}
 \alpha_1 &= a - 2\alpha_2 \\
 3\alpha_2 &= b - 2\alpha_1 - \alpha_3 = b - 2(a - 2\alpha_2) - 4\alpha_3 \\
 &= b - 2a + 4\alpha_2 - 4\alpha_3 \\
 \alpha_2 &= 2a - b + 4\alpha_3 \\
 \alpha_3 &= c - 3\alpha_1 - 4\alpha_2 = c - 3(a - 2\alpha_2) - 4\alpha_2 \\
 &= c - 3a + 2\alpha_2 = c - 3a + 2(2a - b + 4\alpha_3) \\
 &= c + a - 2b + 4\alpha_3 \\
 \alpha_3 &= 2/3b - 1/3a - 1/3c.
 \end{aligned}$$

Since these equations have a solution, T is a spanning set for \mathbb{R}^3 .

3. (a) Is $S = \{(1, 1, 0, 0), (1, -1, 0, 0), (0, 0, 1, -1), (0, 0, -1, 1)\}$ a spanning set for \mathbb{R}^4 ?

Solution: We solve

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 - \alpha_2 \\ \alpha_3 - \alpha_4 \\ -\alpha_3 + \alpha_4 \end{bmatrix}$$

which gives us the system of equations

$$\alpha_1 + \alpha_2 = a \quad \alpha_1 - \alpha_2 = b \quad \alpha_3 - \alpha_4 = c \quad \alpha_4 - \alpha_3 = d$$

and we see that we must have $c = -d$, so the set does not span.

(We see that we would have $\alpha_1 = (a + b)/2$ and $\alpha_2 = (a - b)/2$).

- (b) Is $T = \{1, 1 + x, 1 + x^2\}$ a spanning set for $\mathcal{P}_2(x)$?

Solution: We want to solve

$$a + bx + cx^2 = \alpha_0(1) + \alpha_1(1 + x) + \alpha_2(1 + x^2) = (\alpha_0 + \alpha_1 + \alpha_2) + \alpha_1x + \alpha_2x^2$$

which gives us the system

$$a = \alpha_0 + \alpha_1 + \alpha_2 \quad b = \alpha_1 \quad c = \alpha_2$$

which has the solution

$$\alpha_2 = c \quad \alpha_1 = b \quad \alpha_0 = a - b - c.$$

Thus this is a spanning set.

4. Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subset V$ is a spanning set for V . Show that $T = \{\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_2, \dots, \mathbf{v}_n - \mathbf{v}_{n-1}\}$ is a spanning set for V .

Solution: Suppose we have an element $\mathbf{u} \in V$. Since S is a spanning set for V , this means that \mathbf{u} is a linear combination of elements of S , so there exist b_1, \dots, b_n such that

$$\mathbf{u} = a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n$$

so we want to solve the equation

$$\begin{aligned} a_1 \mathbf{v}_1 + \cdots + a_n \mathbf{v}_n &= \alpha_1 \mathbf{v}_1 + \alpha_2 (\mathbf{v}_2 - \mathbf{v}_1) + \cdots + \alpha_n (\mathbf{v}_n - \mathbf{v}_{n-1}) \\ &= (\alpha_1 - \alpha_2) \mathbf{v}_1 + \cdots + (\alpha_{n-1} - \alpha_n) \mathbf{v}_{n-1} + \alpha_n \mathbf{v}_n \end{aligned}$$

which gives us the system

$$a_1 = \alpha_1 - \alpha_2 \qquad \cdots \qquad a_{n-1} = \alpha_{n-1} - \alpha_n \qquad a_n = \alpha_n$$

and we can solve this to get

$$\begin{aligned} \alpha_n &= a_n & \alpha_{n-1} &= a_{n-1} + \alpha_n = a_{n-1} + a_n \\ \dots & & \alpha_1 &= a_1 + a_2 + a_3 + \cdots + a_n. \end{aligned}$$

Thus $\mathbf{u} \in \text{Span}(T)$. Since \mathbf{u} was an arbitrary vector in V , this means that T spans V .

5. (a) Is $S = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ a linearly independent set?

Solution: Suppose

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a + b + c \\ b + c \\ c \end{bmatrix}.$$

Then we have the system

$$a + b + c = 0 \qquad b + c = 0 \qquad c = 0$$

from which we see that $c = 0$, and thus $b = 0 - c = 0$ and $a = 0 - b - c = 0 - 0 - 0 = 0$. Thus S is linearly independent.

- (b) Is $T = \{(1, 2, 3), (4, 5, 6), (7, 8, 9)\}$ a linearly independent set?

Solution: Suppose

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + c \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} a + 4b + 7c \\ 2a + 5b + 8c \\ 3a + 6b + 9c \end{bmatrix}.$$

Then we have the system

$$a + 4b + 7c = 0 \qquad 2a + 5b + 8c = 0 \qquad 3a + 6b + 9c = 0.$$

Then we have

$$\begin{aligned} a &= -4b - 7c \\ 5b &= -2a - 8c = -2(-4b - 7c) - 8c = 8b + 6c \\ b &= -2c \\ a &= 8c - 7c = c \\ c &= -a/3 - 2b/3 = -c/3 + 4c/3. \end{aligned}$$

In particular we see that if $c = 1, a = 1, b = -2$ then we have a solution to this system.

Alternatively, we can simply notice that

$$\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

and thus the set is not linearly independent, since we can write one element as a linear combination of the others.

- (c) Is $U = \{(3, 7, 5), (2, 4, 2), (1, 3, 1)\}$ a linearly independent set?

Solution: Suppose

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} + b \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} + c \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3a + 2b + c \\ 7a + 4b + 3c \\ 5a + 2b + c \end{bmatrix}.$$

Then we have the system

$$3a + 2b + c = 0 \qquad 7a + 4b + 3c = 0 \qquad 5a + 2b + c = 0.$$

We can subtract the first equation from the third to see $2a = 0$ and thus $a = 0$. Then we have $c = -2b$ and $4b = -3c = 6b$ so $b = 0$ and then $c = 0$. So U is linearly independent.

6. (a) Is $S = \{1 + x, 1 + x^2, x + x^2\}$ a linearly independent set?

Solution: Suppose

$$0 = a(1 + x) + b(1 + x^2) + c(x + x^2) = (a + b) + (a + c)x + (b + c)x^2.$$

Then we have the system

$$0 = a + b \qquad 0 = a + c \qquad 0 = b + c.$$

This tells us that $a = -b$ from the first equation, and thus $b = c$ from the second, and thus $2c = 0$ from the third. Thus $c = 0$, and then $b = 0$ and $a = 0$. So S is linearly independent.

- (b) Is $T = \{1 + x, 1 + x^2, x - x^2\}$ a linearly independent set?

Solution: Suppose

$$0 = a(1 + x) + b(1 + x^2) + c(x - x^2) = (a + b) + (a + c)x + (b - c)x^2.$$

This gives us the system

$$0 = a + b \qquad 0 = a + c \qquad 0 = b - c.$$

This gives us $b = c$, and then the other two equations become the same; so we see that if $c = 1$ then $b = 1, a = -1$ is a solution to the system.

Alternatively, we can notice that

$$(1 + x) - (1 + x^2) = (1 - 1) + x - x^2 = x - x^2$$

so T is not linearly independent because one of the vectors can be written as a linear combination of the others.

(c) Is $U = \{\sin^2, \cos^2, 1\}$ a linearly independent set?

Solution: We don't have an easy way to turn this into a system of linear equations. But we can notice (or recall from class) that $\sin^2 + \cos^2 = 1$. Thus U is not linearly independent, since one vector can be written as a linear combination of the others.

7. (★) Suppose $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent in V , and $T = \{\mathbf{v}_1 + \mathbf{w}, \dots, \mathbf{v}_n + \mathbf{w}\}$ is linearly dependent in V . Show that $\mathbf{w} \in \text{Span}(S)$.

Solution: Since T is linearly dependent, we have constants a_1, \dots, a_n not all zero such that

$$0 = a_1(\mathbf{v}_1 + \mathbf{w}) + \dots + a_n(\mathbf{v}_n + \mathbf{w}).$$

We can rearrange this equation to give

$$-(a_1 + \dots + a_n)\mathbf{w} = a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n.$$

We would like to divide by $-(a_1 + \dots + a_n)$, but we may only do this if we know $a_1 + \dots + a_n \neq 0$, so we need to prove that somehow. So suppose for contradiction that $a_1 + \dots + a_n = 0$. Then our previous equation becomes

$$\mathbf{0} = a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n.$$

Since S is linearly independent, we know that $a_1 = \dots = a_n = 0$; but by hypothesis we know that some a_i is nonzero, which is a contradiction. Thus we must have $a_1 + \dots + a_n \neq 0$.

Then we have the equation

$$\mathbf{w} = \frac{-a_1}{a_1 + \dots + a_n}\mathbf{v}_1 + \dots + \frac{-a_n}{a_1 + \dots + a_n}\mathbf{v}_n.$$

Then we have written \mathbf{w} as a linear combination of vectors in S , so $\mathbf{w} \in \text{Span}(S)$.

8. Prove that a set $S = \{\mathbf{u}, \mathbf{v}\}$ of two vectors is linearly dependent if and only if one is a scalar multiple of the other.

Solution: Suppose S is linearly dependent. Then there is some solution to $\mathbf{0} = a\mathbf{u} + b\mathbf{v}$ where the coefficients are not both zero; without loss of generality assume $a \neq 0$. Then we can write $\mathbf{u} = -b/a\mathbf{v}$ and thus one vector is a scalar multiple of the other.

Conversely, suppose \mathbf{u} is a scalar multiple of \mathbf{v} , that is, suppose there is a scalar a with $\mathbf{u} = a\mathbf{v}$. Then we can write $\mathbf{0} = (-1)\mathbf{u} + a\mathbf{v}$. Since $-1 \neq 0$, we have written $\mathbf{0}$ as a nontrivial linear combination of \mathbf{u} and \mathbf{v} , so S is linearly dependent.