Math 214 Spring 2017
Linear Algebra HW 5 Solutions
Due **Friday**, March 3

For all these problems, justify your answers.

1. Find all solutions to the system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 9 \\ 2x_1 - x_2 + x_3 &= 0 \\ 4x_1 - x_2 + x_3 &= 4. \end{aligned}$$

Solution:

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 2 & -3 & 9 \\ 0 & -5 & 7 & -18 \\ 0 & -9 & 13 & -32 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 2 & -3 & 9 \\ 0 & 1 & -7/5 & 18/5 \\ 0 & -9 & 13 & -32 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|r} 1 & 0 & -1/5 & 9/5 \\ 0 & 1 & -7/5 & 18/5 \\ 0 & 0 & 2/5 & 2/5 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -1/5 & 9/5 \\ 0 & 1 & -7/5 & 18/5 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus the solution is $x_1 = 2, x_2 = 5, x_3 = 1$.

2. Find all solutions to the system

$$\begin{aligned} 2x_1 + 3x_2 - x_3 + 4x_4 &= 1 \\ 3x_1 - x_2 + x_4 &= 1 \\ 3x_1 - 4x_2 + x_3 - x_4 &= 2. \end{aligned}$$

Solution:

$$\left[\begin{array}{rrrr|r} 2 & 3 & -1 & 4 & 1 \\ 3 & -1 & 0 & 1 & 1 \\ 3 & -4 & 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 3/2 & -1/2 & 2 & 1/2 \\ 3 & -1 & 0 & 1 & 1 \\ 3 & -4 & 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 3/2 & -1/2 & 2 & 1/2 \\ 0 & -11/2 & 3/2 & -5 & -1/2 \\ 0 & -17/2 & 5/2 & -7 & 1/2 \end{array}\right]$$

$$\to \left[\begin{array}{rrrr|r} 1 & 3/2 & -1/2 & 2 & 1/2 \\ 0 & 1 & -3/11 & 10/11 & 1/11 \\ 0 & -17/2 & 5/2 & -7 & 1/2 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & -1/11 & 7/11 & 4/11 \\ 0 & 1 & -3/11 & 10/11 & 1/11 \\ 0 & 0 & 2/11 & 8/11 & 14/11 \end{array}\right]$$

$$\to \left[\begin{array}{rrrr|r} 1 & 0 & -1/11 & 7/11 & 4/11 \\ 0 & 1 & -3/11 & 10/11 & 1/11 \\ 0 & 0 & 1 & 4 & 7 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 4 & 7 \end{array}\right].$$

Thus we set the free variable $x_4 = \alpha$ and our set of solutions is

$$\{(1-\alpha, 2-2\alpha, 7-4\alpha, \alpha) = \{(1,2,7,0) + \alpha(-1,-2,-4,1)\}.$$

3. Find all solutions to the system

$$\begin{aligned} 5x_1 + 3x_2 - x_3 &= 3 \\ 3x_1 - 3x_2 + x_3 &= 2 \\ -x_1 + 2x_2 + 4x_3 &= 1 \\ x_1 - x_2 + 3x_3 &= 0. \end{aligned}$$

Solution:

$$\left[\begin{array}{rrr|r} 5 & 3 & -1 & 3 \\ 3 & -3 & 1 & 2 \\ -1 & 2 & 4 & 1 \\ 1 & -1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -1 & 3 & 0 \\ 5 & 3 & -1 & 3 \\ 3 & -3 & 1 & 2 \\ -1 & 2 & 4 & 1 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -1 & 3 & 0 \\ 0 & 8 & -16 & 3 \\ 0 & 0 & -8 & 2 \\ 0 & 1 & 7 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|r} 1 & -1 & 3 & 0 \\ 0 & 1 & 7 & 1 \\ 0 & 8 & -16 & 3 \\ 0 & 0 & -8 & 2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 10 & 1 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & -72 & -5 \\ 0 & 0 & -8 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|r} 1 & 0 & 10 & 1 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & -72 & -5 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 7/2 \\ 0 & 1 & 0 & 11/4 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 13 \end{array}\right].$$

Since the last equation is $0 = 13$ this system is inconsistent and has no solutions.

4. Reduce the following matrix to reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 1 & -10 & -9 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 16 & 14 & -5 \\ 0 & 1 & -10 & -9 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 0 & 29 & 29 & 25 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 16 & 14 & -5 \\ 0 & 1 & -10 & -9 & 5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 29 & 29 & 25 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 & 11 \\ 0 & 1 & 0 & 1 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 54 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -2 & 11 \\ 0 & 1 & 0 & 1 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

5. Reduce the following matrix to reduced row echelon form:

$$\begin{bmatrix} 2 & -3 & -1 & 2 & 3 & 4 \\ 4 & -4 & -1 & 4 & 11 & 4 \\ 2 & -5 & -2 & 2 & -1 & 9 \\ 0 & 2 & 1 & 0 & 4 & -5 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 2 & -3 & -1 & 2 & 3 & 4 \\ 4 & -4 & -1 & 4 & 11 & 4 \\ 2 & -5 & -2 & 2 & -1 & 9 \\ 0 & 2 & 1 & 0 & 4 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -3/2 & -1/2 & 1 & 3/2 & 2 \\ 4 & -4 & -1 & 4 & 11 & 4 \\ 2 & -5 & -2 & 2 & -1 & 9 \\ 0 & 2 & 1 & 0 & 4 & -5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -3/2 & -1/2 & 1 & 3/2 & 2 \\ 0 & 2 & 1 & 0 & 5 & -4 \\ 0 & -2 & -1 & 0 & -4 & 5 \\ 0 & 2 & 1 & 0 & 4 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -3/2 & -1/2 & 1 & 3/2 & 2 \\ 0 & 2 & 1 & 0 & 5 & -4 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -3/2 & -1/2 & 1 & 3/2 & 2 \\ 0 & 1 & 1/2 & 0 & 5/2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -3/2 & -1/2 & 1 & 3/2 & 2 \\ 0 & 1 & 1/2 & 0 & 5/2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1/4 & 1 & 21/4 & -1 \\ 0 & 1 & 1/2 & 0 & 5/2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/4 & 1 & 0 & -25/4 \\ 0 & 1 & 1/2 & 0 & 0 & -9/2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

6. Compute

$$\begin{bmatrix} 6 & 3 & 10 \\ 4 & 8 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1.5 \\ 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 28 & 38 \\ 41 & 40 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -1 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 & 3 & -1 \\ 5 & -2 & -1 & 1 \\ -1 & 2 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 12 & -8 & 0 & -4 \\ 2 & 4 & -5 & 7 \end{bmatrix}$$

7. Compute:

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 1 \\ -1 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -9 & -8 \\ -6 & -4 & 5 \\ 5 & 0 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 1 \\ -1 & 6 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -7 & 3 & -1 \\ 6 & -1 & -4 \\ -11 & 6 & 4 \end{bmatrix}$$

8. ($\star$) Let A be a fixed $m \times n$ matrix, and $\mathbf{x}$ a column vector of unknowns in $\mathbb{R}^n$. If U is the subset of $\mathbb{R}^n$ of all solutions to the equation $A\mathbf{x} = \mathbf{0}$, prove that U is a subspace of $\mathbb{R}^n$.

Solution: The set of solutions to $A\mathbf{x} = \mathbf{0}$ is a subset of $\mathbb{R}^n$, and thus by the subspace lemma we need to check three things.

(a) $\mathbf{0}$ is clearly in U since $A\mathbf{0} = \mathbf{0}$.

(b) If $r \in \mathbb{R}$ and $\mathbf{x} \in U$, then $A(r\mathbf{x}) = r(A\mathbf{x}) = r\mathbf{0} = \mathbf{0}$, so $r\mathbf{x} \in U$. Thus U is closed under scalar multiplication.

(c) If $\mathbf{x}, \mathbf{y} \in U$, then $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, so $\mathbf{x} + y \in U$. Thus U is closed under addition.

So by the subspace theorem, U is a subspace of $\mathbb{R}^n$.