Math 214 Spring 2017
Linear Algebra HW 6 Solutions
Due Friday, March 17

For all these problems, justify your answers.

1. Let $A = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$. Find a basis for the rowspace, columnspace, and nullspace of A.

Solution:

$$\begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 14 & 1 & -4 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -10/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then a basis for the rowspace is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -10/7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2/7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

A basis for the columnspace is

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \right\}.$$

The matrix has trivial nullspace, so a basis for the nullspace is $\{\}$.

2. Let $B = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$. Find a basis for the rowspace, columnspace, and nullspace of B.

Solution:

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & -5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & -5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus a basis for the rowspace is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \right\}$$

A basis for the columnspace is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix} \right\}.$$

To find the kernel, we set the third column as the free variable, and the kernel is $\{(-2\alpha, 0, \alpha)\}$. Thus a basis is $\{(-2, 0, 1)\}$.

3. Use Gaussian elimination to find a basis for the span of $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$.

   Solution: The simplest approach is to make each of these vectors a *row* of a matrix, and then do row reduction.

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \\ 3 & -2 & 5 \\ 2 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Thus

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

   is a basis for the span.

   Alternatively, you could make these vectors the columns of a matrix, and find a basis for the columnspace.

$$\begin{bmatrix} 1 & -2 & 3 & 2 \\ -1 & 2 & -2 & -1 \\ 2 & -4 & 5 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   Since the first and third columns have leading 1s, the first and third vectors form a basis for the span. Thus our basis is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}.$$

4. For each of the following systems of equations, is there a solution? You don't need to find the solution if it exists, but justify your answer. (Hint: think about the column space).

(a)

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}?$$

Solution: It's easy to see that $(1, 1, 1)$ is not in the span of the column vectors, since anything in the span of the column vectors has a middle coordinate of 0. So the system has no solutions.

(b)

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 1 \end{bmatrix}?$$

Solution: The columns of the matrix form a basis for $\mathbb{R}^3$, so $(3, 17, 1)$ is in their span. Thus there is a solution to this system.

5. ($\star$) Find the inverse of $\begin{bmatrix} 0 & -1 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 1 & -2 & 3 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ or prove it is not invertible.

Solution:

$$\left[\begin{array}{cccc|cccc} 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & -2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & -2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & -2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 5 & -6 & 2 & 0 & 1 & -2 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & -2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 5 & -6 & 2 & 0 & 1 & -2 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 5 & 1 & -2 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -5 & -1 & 2 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 3 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & -6 & -1 & 2 & 0 \\ 0 & 0 & 1 & -2 & -5 & -1 & 2 & 0 \\ 0 & 0 & 0 & 3 & 11 & 2 & -4 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 3 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & -6 & -1 & 2 & 0 \\ 0 & 0 & 1 & -2 & -5 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 11/3 & 2/3 & -4/3 & 1/3 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -13/3 & -1/3 & 5/3 & -2/3 \\ 0 & 1 & 0 & 0 & 4/3 & 1/3 & -2/3 & 2/3 \\ 0 & 0 & 1 & 0 & 7/3 & 1/3 & -2/3 & 2/3 \\ 0 & 0 & 0 & 1 & 11/3 & 2/3 & -4/3 & 1/3 \end{array}\right].$$

Thus we have

$$A^{-1} = \begin{bmatrix} -13/3 & -1/3 & 5/3 & -2/3 \\ 4/3 & 1/3 & -2/3 & 2/3 \\ 7/3 & 1/3 & -2/3 & 2/3 \\ 11/3 & 2/3 & -4/3 & 1/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -13 & -1 & 5 & -2 \\ 4 & 1 & -2 & 2 \\ 7 & 1 & -2 & 2 \\ 11 & 2 & -4 & 1 \end{bmatrix}.$$

6. Find the inverse of $\begin{bmatrix} 3 & 2 & 1 & 5 \\ 2 & 4 & 3 & 8 \\ -1 & 2 & 5 & 4 \\ 4 & 8 & 9 & 17 \end{bmatrix}$ or prove it is not invertible.

Solution: The sum of the first three rows is the fourth row. Thus the rows are linearly dependent, and the inverse does not exist.

By Gaussian elimination:

$$\left[\begin{array}{cccc|cccc} 3 & 2 & 1 & 5 & 1 & 0 & 0 & 0 \\ 2 & 4 & 3 & 8 & 0 & 1 & 0 & 0 \\ -1 & 2 & 5 & 4 & 0 & 0 & 1 & 0 \\ 4 & 8 & 9 & 17 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & -2 & -5 & -4 & 0 & 0 & -1 & 0 \\ 3 & 2 & 1 & 5 & 1 & 0 & 0 & 0 \\ 2 & 4 & 3 & 8 & 0 & 1 & 0 & 0 \\ 4 & 8 & 9 & 17 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & -2 & -5 & -4 & 0 & 0 & -1 & 0 \\ 0 & 8 & 16 & 17 & 1 & 0 & 3 & 0 \\ 0 & 8 & 13 & 16 & 0 & 1 & 2 & 0 \\ 0 & 16 & 29 & 33 & 0 & 0 & 4 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & -2 & -5 & -4 & 0 & 0 & -1 & 0 \\ 0 & 8 & 16 & 17 & 1 & 0 & 3 & 0 \\ 0 & 0 & -3 & -1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -3 & -1 & -2 & 0 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & -2 & -5 & -4 & 0 & 0 & -1 & 0 \\ 0 & 8 & 16 & 17 & 1 & 0 & 3 & 0 \\ 0 & 0 & -3 & -1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right].$$

Thus the matrix is not full rank, and has non-zero kernel, and thus is singular.

7. Let $L : U \to V$ be a linear transformation. Prove that $\ker(L)$ is a subspace of U. (Hint: see problem 8 from homework 5).

Solution: We need to check three things for the subspace theorem.

(a) We see that $L(\mathbf{0}) = L(0 \cdot \mathbf{0}) = 0L(\mathbf{0}) = \mathbf{0}$, so $\mathbf{0} \in \ker(L)$.

(b) Suppose $r \in \mathbb{R}, \mathbf{u} \in \ker(L)$. Then $L(r\mathbf{u}) = rL(\mathbf{u}) = r\mathbf{0} = \mathbf{0}$, so $r\mathbf{u} \in \ker(L)$.

(c) Suppose $\mathbf{u}, \mathbf{v} \in \ker(L)$. Then $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u})+L(\mathbf{v}) = \mathbf{0}+\mathbf{0} = \mathbf{0}$, so $\mathbf{u}+\in \ker(L)$.

Thus by the subspace theorem, $\ker(L)$ is a subspace of U.

8. ($\star$) Let $\mathcal{C}([a, b], \mathbb{R})$ be the space of continuous functions defined on the closed interval $[a, b]$. Prove that the function from $\mathcal{C}([a, b], \mathbb{R})$ to $\mathbb{R}$ given by $f \mapsto \int_a^b f(t)\, dt$ is a linear transformation. What is the kernel of this transformation?

Solution: If $r \in \mathbb{R}$ and $f \in \mathcal{C}([a, b], \mathbb{R})$, then

$$\int_a^b (rf)(t)\, dt = \int_a^b rf(t)\, dt = r \int_a^b f(t)\, dt$$

and if $f, g \in \mathcal{C}([a, b], \mathbb{R})$, then

$$\int_a^b (f+g)(t)\, dt = \int_a^b f(t) + g(t)\, dt = \int_a^b f(t)\, dt + \int_a^b g(t)\, dt.$$

Thus by definition this map is a linear transformation.

The kernel is the set of all functions f such that $\int_a^b f(t)\, dt = 0$. We can think of this as the set of all functions whose average value over the interval $[a, b]$ is zero.