Math 214 Spring 2017
Linear Algebra HW 7 Solutions
Due *Wednesday*, March 22

For all these problems, justify your answers.

1. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$L((x,y,z)) = \begin{bmatrix} x+y+z \\ 3x-2y+z \\ 2z \end{bmatrix}.$$

   (a) Prove that L is a linear transformation.

   (b) Find a basis for the kernel and for the image.

2. Let $\mathcal{F}(\mathbb{R},\mathbb{R})$ be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Define $E_0 : \mathcal{F}(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ to be the function given by $E_0(f) = f(0)$.

   (a) Prove that E_0 is a linear transformation.

   (b) What is the kernel of E_0? What is the image?

   Solution:

   (a) We see that $E_0(f+g) = (f+g)(0) = f(0)+g(0) = E_0(f)+E_0(g)$. We also see that $E_0(rf) = (rf)(0) = rf(0) = rE_0(f)$. Thus E_0 is a linear transformation by definition.

   (b) $E_0(f) = 0$ if and only if $f(0) = 0$. Thus the kernel of E_0 is $\{f : \mathbb{R} \to \mathbb{R} | f(0) = 0\}$. The image of E_0 is all reals, since for any r we can take the constant function $f(x) = r$ and we have $E_0(f) = f(0) = r$. (Thus E_0 is onto, but not one-to-one).

3. Let $P_Z : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection map onto the xy plane, given by $P(x,y,z) = (x,y,0)$.

   (a) Prove that P_z is a linear transformation.

   (b) Find bases for the kernel and image of P_z.

   (c) Find a matrix for P_z with respect to the standard basis.

   (d) Prove that $P_z(P_z(\mathbf{u})) = P_z(\mathbf{u})$ for any $\mathbf{u} \in \mathbb{R}^3$. Linear transformations with this property are called *projections* and we will revisit them later. (They are also sometimes called *idempotent* if you're feeling particularly fancy).

Solution:

(a) $P(r(x,y,z)) = P(rx,ry,rz) = (rx,ry,0) = r(x,y,0) = rP(x,y,z)$.
$P((x_1,y_1,z_1)+(x_2,y_2,z_2)) = P(x_1+x_2,y_1+y_2,z_1+z_2) = (x_1+x_2,y_1+y_2,0) = (x_1,y_1,0)+(x_2,y_2,0) = P(x_1,y_1,z_1)+P(x_2,y_2,z_2)$. Thus by defnition P_z is a linear transformation.

(b) $P_z(x,y,z) = (x,y,0) = \mathbf{0}$ when $x = y = 0$. Thus $\ker(P_z) = \{(0,0,z)\}$. Thus a basis for the kernel is $\{(0,0,1)\}$.
The image of P_z is $\{(x,y,0)\}$. So a basis is $\{(1,0,0),(0,1,0)\}$.

(c) We have $P_z(1,0,0) = (1,0,0), P_z(0,1,0) = (0,1,0), P_z(0,0,1) = (0,0,0)$. Thus a matrix for P_z is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(d) $P_z(P_z(x,y,z)) = P_z(x,y,0) = (x,y,0) = P_z(x,y,z)$.

4. (a) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. If $L((1,2)) = (-2,3)$ and $L((1,-1)) = (5,2)$, what is $L((7,5))$?

(b) $E = \{(1,1,0),(1,0,1),(0,1,1)\}$ is a basis for $\mathbb{R}^3$. Take the vector $\mathbf{u}$ represented by $(2,3,4)$ in the standard basis, and calculate $[\mathbf{u}]_E$.

(c) $F = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$ is a basis for $\mathcal{P}_3(x)$.
Calculate $[3+5x-2x^2+x^3]_F$.

5. Let $L : \mathbb{R} \to \mathbb{R}$ be a linear transformation. Prove that there is some real number $r \in \mathbb{R}$ such that $L(x) = rx$ for all $x \in \mathbb{R}$. (In other words, any linear transformation from $\mathbb{R}$ to $\mathbb{R}$ is given by multiplication by a scalar).

Solution: Set $r = L(1)$. Then if $x \in \mathbb{R}$, we have $L(x) = xL(1) = xr = rx$.

Alternatively, you can observe that a linear transformation from $\mathbb{R}$ to $\mathbb{R}$ is given by multiplication by a 1×1 matrix, which is just a scalar.

6. ($\star$) Let $L : U \to V$ be a linear transformation, and let $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for U. Prove that $L(U) = \operatorname{Span}(L(E))$. That is, prove the image of L is just the span of the image of E under L.

Solution:

The first is mutual subset inclusion. Suppose $\mathbf{v} \in \operatorname{Span}(L(E))$. Then we can write

$$\begin{aligned} \mathbf{v} &= a_1 L(\mathbf{e}_1) + \cdots + a_n L(\mathbf{e}_n) = L(a\mathbf{e}_1) + \cdots + L(a\mathbf{e}_n) \\ &= L(a\mathbf{e}_1 + \cdots + a\mathbf{e}_n) \in L(U). \end{aligned}$$

Thus $\operatorname{Span}(L(E)) \subseteq L(U)$.

Conversely, let $\mathbf{v} \in L(U)$. Then we can write $\mathbf{v} = L(\mathbf{u})$ for some $\mathbf{u} \in U$. Since E is a basis for U, we can write

$$\begin{aligned} \mathbf{u} &= a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n \\ L(\mathbf{u}) &= L(a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n) = L(a_1\mathbf{e}_1) + \cdots + L(a_n\mathbf{e}_n) \\ &= a_1 L(\mathbf{e}_1) + \cdots + a_n L(\mathbf{e}_n) \in \operatorname{Span}(L(E)). \end{aligned}$$

Thus $L(U) \subseteq \operatorname{Span}(L(E))$, so $L(U) = \operatorname{Span}(L(E))$.

Bonus: Don't hand in, because we won't get here by Monday, but practice for test on Friday:

($\star$) Let $U = \mathcal{P}_3(x)$, and define a linear map $D : U \to U$ by $D(f(x)) = f'(x)$. Let $E = \{1, x, x^2, x^3\}$ be a basis for U.

1. What are the kernel and image of D?

2. Find a matrix for D with respect to E and E.

Solution:

1. $\ker(D)$ is the set of all polynomials whose derivative is zero, which is just the set of constants. Thus $\ker(D) = \{a_0 + 0x + 0x^2 + 0x^3 : a_0 \in \mathbb{R}\} = \{a_0 : a_0 \in \mathbb{R}\}$.

   The image of D is the set of all polynomials of degree 2 or less. If $a_0 + a_1x + a_2x^2$ is a degree two polynomial, then $a_0x + a_1/2x^2 + a_2/3x^3 \in \mathcal{P}_3(x)$ with $D(a_0x + a_1/2x^2 + a_2/3x^3) = a_0 + a_1x + a_2x^2$.

2. We have

$$[D(1)]_E = [0]_E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad [D(x)]_E = [1]_E = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[D(x^2)]_E = [2x]_E = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \qquad [D(x^3)]_E = [3x^2]_E = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}.$$

   Thus the matrix with respect to E is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$