Math 214 Spring 2017
Linear Algebra HW 8
Due Friday March 31

For all these problems, justify your answers.

1. Let $L : U \to V$ be a linear transformation of vector spaces. Prove that L is one-to-one if and only if $\ker(L) = \{\mathbf{0}\}$.

2. Let $f(x, y, z) = (2x - y, y + z, z + x)$ and $g(a, b, c) = (a + b - c, a + 2b - 2c, -a - b + 2c)$. Prove that g is the inverse of f.

3. Prove that inverses are unique. That is, let $f : U \to V$, and let $g, h : V \to U$ such that $g(f(\mathbf{u})) = \mathbf{u} = h(f(\mathbf{u}))$ and $f(g(\mathbf{v})) = \mathbf{v} = f(h(\mathbf{v}))$. Prove that $g = h$.

4. Let $L : U \to V$ be a linear transformation.
   (a) If $\dim U > \dim V$, prove that L is not injective. (There is "too much" in U to fit it all in V without repeating).
   (b) If $\dim U < \dim V$, prove that L is not surjective. (There is "not enough" in U to cover all of V).

5. (a) Suppose $L : \mathbb{R}^3 \to \mathcal{P}_2(x)$ is linear and surjective. Prove it is an isomorphism.
   (b) Suppose $T : \mathcal{P}_5(x) \to \mathbb{R}^6$ is linear with trivial kernel. Prove it is an isomorphism.

Fun things to think about:

- Is $f(x) = x^2$ an inverse of $g(x) = \sqrt{x}$? Why or why not? Does it depend on information I haven't given you?
- Find counterexamples to the converse of the statements in problem 4. That is, find a function $L : U \to V$ where L is not injective, but $\dim U < \dim V$. And find a function $L : U \to V$ where L is not surjective, but $\dim U > \dim V$.