# Math 214 Spring 2017
# Linear Algebra HW 8 Solutions
# Due Friday March 31

For all these problems, justify your answers.

1. Let $L : U \to V$ be a linear transformation of vector spaces. Prove that $L$ is one-to-one if and only if $\ker(L) = \{\mathbf{0}\}$.

   **Solution:** Suppose $L$ is one-to-one, and suppose $\mathbf{u} \in \ker(L)$, that is, $L(\mathbf{u}) = \mathbf{0}$. We know that $L(\mathbf{0}) = \mathbf{0}$, so by definition of one-to-one we know that $\mathbf{u} = \mathbf{0}$. Thus $\ker(L) = \{\mathbf{0}\}$.

   Conversely, suppose $\ker(L) = \{0\}$, and supppose $\mathbf{u}, \mathbf{v} \in U$ with $L(\mathbf{u}) = L(\mathbf{v})$. Then by linearity we have

   $$\mathbf{0} = L(\mathbf{u}) - L(\mathbf{v}) = L(\mathbf{u} - \mathbf{v})$$

   so $\mathbf{u} - \mathbf{v} \in \ker(U) = \{\mathbf{0}\}$. Thus $\mathbf{u} - \mathbf{v} = \mathbf{0}$ and so $\mathbf{u} = \mathbf{v}$.

2. Let $f(x, y, z) = (2x - y, y + z, z + x)$ and $g(a, b, c) = (a + b - c, a + 2b - 2c, -a - b + 2c)$. Prove that $g$ is the inverse of $f$.

   **Solution:**

   $$\begin{aligned} g(f(x,y,z)) &= g(2x - y, y + z, z + x) \\ &= \begin{bmatrix} (2x-y) + (y+z) - (z+x) \\ (2x-y) + 2(y+z) - 2(z+x) \\ -(2x-y) - (y+z) + 2(z+x) \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \\ f(g(a,b,c) &= f(a + b - c, a + 2b - 2c, -a - b + 2c) \\ &= \begin{bmatrix} 2(a+b-c) - (a+2b-2c) \\ (a+2b-2c) + (-a-b+2c) \\ (-a-b+2c) + (a+b-c) \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \end{aligned}$$

3. Prove that inverses are unique. That is, let $f : U \to V$, and let $g, h : V \to U$ such that $g(f(\mathbf{u})) = \mathbf{u} = h(f(\mathbf{u}))$ and $f(g(\mathbf{v})) = \mathbf{v} = f(h(\mathbf{v}))$. Prove that $g = h$.

   **Solution:** Let $\mathbf{v} \in V$. Then

   $$g(\mathbf{v}) = h(f(g(\mathbf{v})) = h(\mathbf{v}).$$

   Thus $g(\mathbf{v}) = h(\mathbf{v})$ for every $\mathbf{v} \in V$, so $g = h$.

4. Let $L : U \to V$ be a linear transformation.

(a) If $\dim U > \dim V$, prove that $L$ is not injective. (There is "too much" in $U$ to fit it all in $V$ without repeating).

(b) If $\dim U < \dim V$, prove that $L$ is not surjective. (There is "not enough" in $U$ to cover all of $V$).

**Solution:**

(a) We know that $L(U) \subset V$ so $\dim L(U) \leq \dim V$. By the Rank-Nullity Theorem we know that $\dim U = \dim L(V) + \dim \ker(L)$, which we can rewrite as $\dim \ker(L) = \dim U + \dim L(V) \geq \dim(U) - \dim(V) > 0$. Thus the kernel is nontrivial, so the function is not injective.

(b) By the Rank-Nullity Theorem, we know that $\dim U = \dim \ker(L) + \dim L(U) \geq \dim L(U)$. Thus $\dim L(U) \leq \dim(U) < \dim V$ so $L(U) \neq V$.

5. (a) Suppose $L : \mathbb{R}^3 \to \mathcal{P}_2(x)$ is linear and surjective. Prove it is an isomorphism.

   (b) Suppose $T : \mathcal{P}_5(x) \to \mathbb{R}^6$ is linear with trivial kernel. Prove it is an isomorphism.

**Solution:**

(a) We just need to prove that $L$ is injective, that is, that it has trivial kernel. By rank-nullity, we know that $\dim \mathbb{R}^3 = \dim \mathcal{P}_2(x) + \dim \ker(L)$. But $\dim \mathbb{R}^3 = 3 = \dim \mathcal{P}_2(x)$, so we have $3 = 3 + \dim \ker(L)$, so $\dim \ker(L) = 0$ and the kernel is trivial.

Thus $L$ is injective, and since it is also surjective, it is an isomorphism.

(b) $T$ has trivial kernel, so it is injective. We just need to prove that it is surjective. We know that $\dim \mathcal{P}_5(x) = 6 = \dim \mathbb{R}^6$, and by rank-nullity theorem we have $\dim \mathcal{P}_5(x) = \dim T(\mathcal{P}_5(x)) + \dim \ker(T)$. But the kernel is trivial, so $\dim \ker(T) = 0$, and thus we have $6 = \dim T(\mathcal{P}_5(x))$. Thus $T(\mathcal{P}_5(x))$ is a six-dimensional subspace of a six-dimensional space, and hence $T(\mathcal{P}_5(x)) = \mathbb{R}^6$. So $T$ is surjective, and thus an isomorphism.

Fun things to think about:

- Is $f(x) = x^2$ an inverse of $g(x) = \sqrt{x}$? Why or why not? Does it depend on information I haven't given you?

  **Solution:** This is actually quite tricky/surprising, because it totally depends on what you think the domain and codomain are.

  If we think of $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ as a function whose domain and codomain are the *non-negative reals*, then $x^2$ is invertible and its inverse is $\sqrt{x}$.

  However, we normally think of $f$ as being a function with domain and codomain equal to $\mathbb{R}$. Then $f$ is neither injective (since $f(1) = f(-1)$) nor surjective (since $f(x) = -1$ has no solution).

  Or we could think of it as a function $\mathbb{R} \to \mathbb{R}^{\geq 0}$, in which case it is surjective but not injective.

- Find counterexamples to the converse of the statements in problem 4. That is, find a function $L : U \to V$ where $L$ is not injective, but $\dim U < \dim V$. And find a function $L : U \to V$ where $L$ is not surjective, but $\dim U > \dim V$.

  **Solution:** Let $L : \mathbb{R} \to \mathbb{R}^3$ be given by $L(x) = (0, 0, 0)$. Then $L$ is not injective, but clearly $\dim \mathbb{R} < \dim \mathbb{R}^3$.

  Similarly, $T : \mathbb{R}^2 \to \mathbb{R}$ given by $T(x, y) = 0$. This is not surjective, but $\dim \mathbb{R}^2 > \dim \mathbb{R}$.